Operations Research, Jacobs University, Fall 2022 — Session 27

Prof. Sören Petrat — Dec. 6, 2022

Final exam: close to HW/midterm problems, with slight variations and slightly different numbers.

Summary of class content:

1. Introduction

↳ What is OR?
- Scientific/mathematical approach to management decisions.
- Optimization applied to industrial, logistical, organizational problems.
- In class we have developed lots of tools for various problems from different domains.

↳ General OR workflow: Def. problem, gather data, formulate math problem, solve it (then sensitivity analysis and recommendation/implementation)

2. Linear Programming

Central important method of OR ($\frac{1}{2}$ to $\frac{2}{3}$ of this class).

Lots of problems can be recast as or approximated by LP problems!

2.1 Graphical Solutions

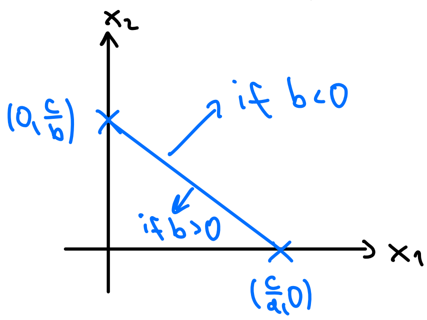
↳ Only works for simple LP problems (2 or 3 decision variables)

↳ For us it was more a tool to understand possible shapes of the feasible region (for LP but also nonlinear problems) and the simplex method

Tips for drawing the feasible region:

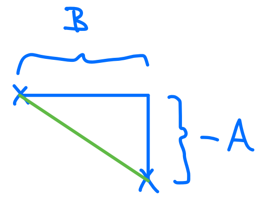
- Constraint $ax_1 + bx_2 \leq c$

  $\Rightarrow$ draw line $ax_1 + bx_2 = c$: it goes through $(0, \frac{c}{b})$ and $(\frac{c}{a}, 0)$



- Objective fct. $Z = Ax_1 + Bx_2$

  $\Rightarrow$ draw line with slope $-\frac{A}{B}$:



Key points:
- Practice finding optimal solutions graphically
- Be aware of different possible shapes of feasible region
- Central role of CPF solutions

2.2 Standard Form of LP Problems

↳ Convention: Minimize $Z = c^T x$ $(c \in \mathbb{R}^m)$,

subject to $Ax \leq b$ $(A \in Mat(n \times m), b \in \mathbb{R}^n)$, and $x \geq 0$.

(n constraints, m decision variables, usually $m > n$)

Main result: If this has optimal solutions, a $\underbrace{\text{basic solution}}$ is among them.

This is the foundation of the simplex method! at least one component $= 0$

2.3 The Simplex Method

↳ Most fundamental algorithm in OR:
- Start with basic feasible sol.
- Choose leaving and entering variables as discussed
- Repeat until no further improvement possible or some problem occurs (e.g., unbounded feasible region)

↳ Practice using a simplex tableau to solve LP problems

↳ But: in practice (large-scale problems), the simplex algorithm is used on computers.
There are many free and commercial software packages available.
We used pyomo because:
- it is integrated in python, which is widely used, free, and has many packages for specialized tasks
- fast, easy to use

⇒ Study how to read pyomo programs and their output!

2.4 The Dual LP Problem

Starting from the standard form above, it is: Minimize $b^T y$,
subject to $A^T y \geq c$ and $y \geq 0$.

↳ Math results:
- $c^T x \leq b^T y$, where x is sol. to primal problem, y to dual problem (weak duality)
- Primal has optimal sol. ⟺ Dual has optimal sol.
  In this case $c^T x = b^T y$ (strong duality)

↳ Applications of dual: • $y_1, \dots, y_m$ = shadow prices = changes of profit (per unit capacity) at current operating conditions

• Value of company (in terms of operational profit) = all ressources valued at shadow prices

2.5 Transportation Problems

Very important for logistics!

↳ An LP problem with very specific constraints:

Minimize $Z = \sum_{ij} c_{ij} x_{ij}$,

subject to $\sum_j x_{ij} = s_i$ (supply rule), $\sum_i x_{ij} = d_j$ (demand rule), and $x_{ij} \geq 0$.

- If $\sum_i s_i = \sum_j d_j$ (supply = demand) then there are feasible sol.
- Integer property

Study how to deal with extra difficulties using dummy sources/sinks/variables.

2.6 Network Optimization

Ubiqitous in applications!

Problem types: • shortest path

• minimum spanning tree

• maximum flow

Study the algorithms to solve these problem types by hand

Overarching framework: Minimum cost flow problem:

Minimize $z = \sum_{ij} c_{ij} x_{ij}$

subject to $\underbrace{\sum_j x_{ij}}_{\text{outflow}} - \underbrace{\sum_j x_{ji}}_{\text{inflow}} = b_i$, and $0 \le x_{ij} \le u_{ij}$ (capacity constraint)

(Extra example: project management (critical path, activity crashing).)

3. Further Optimization Techniques

From here on, we go beyond LP framework

3.1 Dynamic Programming

↳ Very useful general strategy for many applications

↳ General principle: divide problem into different stages; in each stage an optimal policy decision needs to be made

Solution technique:

- We start at the end, and keep track of the optimal costs
- $f_i(s, x_i)$ = cost of optimal route starting at s (stage $i-1$), going through x_i (at stage i), and optimal from then on
  $= c_{s,x_i} + \underbrace{f^*_{i+1}(x_i)}_{\text{or variations thereof, depending on problem context}}$
- $f_i^*(s) = \min_{x_i} f_i(s, x_i)$, the minimum x_i^* is the optimal policy decision (in stage i)

Study how to set this up and solve it with an $\begin{array}{c} \quad x_i \rightarrow \\ s \\ \downarrow \end{array}$ table.

Probabilistic Dynamic Programming

↳ here: minimize <u>expected</u> costs (although a more comprehensive analysis might take other factors into account, e.g., variance)

↳ probabilities enter into $f_i(s, x_i)$ function

3.2 Decision Analysis

This is another important application:

- How to make decisions when consequences are uncertain?
- How to use $\underbrace{\text{additional information}}_{\text{experimentation}}$ (which comes at a cost) to improve decisions?
- As before, we only consider expected profit here.

Key tools: • Bayes' rule: $\mathbb{P}(A \mid B) = \dfrac{\mathbb{P}(B \mid A)\,\mathbb{P}(A)}{\mathbb{P}(B)}$

• Decision trees

3.3 Inventory Theory

Important practical problem!

We studied variations of the EOQ model:

- Basic version: setup cost K, unit cost c, holding cost h, continuous withdrawal rate d, no shortages. Then $Q^* = \sqrt{\frac{2dK}{h}}$ is the optimal order quantity.

(You don't need to memorize the formula, but you need to know how to derive it.)

- Planned shortages: penalty p for shortages. Then $Q^* = \sqrt{\frac{2dK}{h}}\sqrt{\frac{h+p}{p}}$.

Further models:
- Periodic review: demand r_i in period i
  ↳ use dynamic programming
- Perishable products, single period, with <u>stochastic</u> demand: minimize (as before) <u>expected</u> cost
  ↳ Usually one uses a continuous probability distribution. Then $\Phi(y^*) = \frac{p-c}{p+h}$, where y^* = optimal order quantity, Φ = cumulative distribution. $\Phi(y^*)$ = optimal service level = probability that demand is satisfied.
  ↳ For exponential probability distribution we find specifically:
  $y^* = \mu \ln \frac{p+h}{c+h}$

3.4 Nonlinear Programming

Vast field, we just mention some of the problems that can occur (and that we need to be aware of in applications!) and summarize some techniques.

Be aware what can happen if:
- constraints nonlinear, objective fct. linear
- constraints linear, objective fct. nonlinear

Important practical problem (specifically for nonlinear solvers such as ipopt):
local vs. global max./min.

Summary:

Most essential skills we learned that are relevant for the exam:

- Solve LP problems with the graphical method
- Solve LP problems with the simplex method
- Know how to read and interpret a pyomo program
- Know what the dual LP problem is and what it means
- Know how to solve the different types of network optimization problems we discussed
- Solve dynamic programming problems with an "$s - x_i$ table"
- Set up decision trees (using Bayes' rule)
- Derive the basic EOQ model, be aware of the generalizations we discussed
- Know about some of the difficulties of nonlinear programming, solve nonlinear programming problems with the graphical method

All homework problems are relevant for the final exam, with the following exceptions:

- HW 2: only Problem 3 is suitable, not Problems 1 or 2
- HW 10: Problem 2 only in shorter form, e.g., with A_1, A_2, S_1, S_2 only
- HW 11: Problem 1 together with a derivation of the EOQ formula

  Problem 2 (b): no need to memorize the formula
- HW 12: Problem 1 good, but in an exam I would provide the formula $\Phi(y^*) = \frac{p-c}{p+h}$

  Problem 2 good

  Problem 3 not suitable for exam

For more exercises: See the practice exams and solutions on the website.

Final exam organization:

- Final exam on Tue., Dec. 20, 16:00-18:00.
- No notes, no other aides (calculators etc) are permitted.
- Grading will be done via gradescope. Use this to check your exam grading and ask for regrading if necessary.