Stochastic Methods Lab, Jacobs University Bremen, Fall 2022 — Session 5, Sep. 21, 2022

Prof. Dr. Soeren Petrat

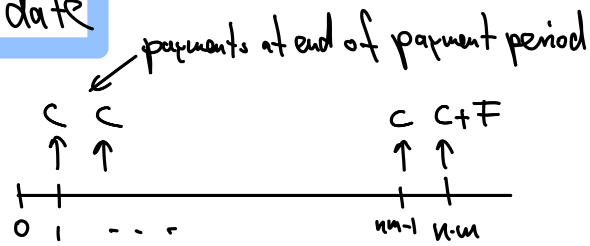
1.3 Bonds

Bond issuer (borrower) makes regular payments and a final payment to bond holder (lender, buyer).

↳ usually for long-term debts, e.g., issued by governments (but also companies)

↳ bonds are fully repaid at maturity date

Cashflow for level-coupon bond:



$$\text{present value} = \text{price} = P = \sum_{i=1}^{n\cdot m} \frac{C}{(1+\frac{r}{m})^i} + \frac{F}{(1+\frac{r}{m})^{nm}}$$

where:
- C = coupon payments
- r = interest rate
- F = par value
- n = # of periods (usually years)
- m = # payments per period
- with $C = \frac{Fc}{m}$, c = coupon rate

$$\Rightarrow P = F\left(\sum_{i=1}^{nm} \frac{c/m}{(1+\frac{r}{m})^i} + \frac{1}{(1+\frac{r}{m})^{nm}}\right)$$

- $P, C (\text{or } c), F, n, m$ determine the "bond contract"; given these values, the r = IRR = yield to maturity

Ex.: $\underbrace{20}_{n}$ year, $\underbrace{9\%\text{ bond}}_{\text{coupon rate}}$, $\underbrace{\text{BEY}}_{\text{"bond equivalent yield"},\ m=2}$, interest rate $r = 8\%$

$$\text{price } P = F\left(\sum_{i=1}^{40} \frac{0.09/2}{1.04^i} + \frac{1}{(1.04)^{40}}\right) = 1.099 \cdot F$$

$\Rightarrow$ this bond sells at 109.9% of par

e.g., par value $F = 1000\$ \Rightarrow P = 1099\$$ and $C = 45\$$.

Using the geometric series, we find (let's do $m=1$ here):

$$P = F\left(c \underbrace{\sum_{i=1}^{n} \frac{1}{(1+r)^i}} + \frac{1}{(1+r)^n}\right)$$

$$\sum_{i=0}^{N} x^i = \frac{1-x^{N+1}}{1-x}$$

$$= -1 + \frac{1-(1+r)^{-n-1}}{1-(1+r)^{-1}} = \frac{-1+(1+r)^{-1}+1-(1+r)^{-n-1}}{1-(1+r)^{-1}} = \frac{(1+r)^{-1}-(1+r)^{-n-1}}{1-(1+r)^{-1}} = \frac{1-(1+r)^{-n}}{r}$$

$$\left(\frac{1+r}{1+r}\right)$$

$$= F\left(\frac{c}{r}\left(1-(1+r)^{-n}\right) + (1+r)^{-n}\right)$$

$$= F\left(\frac{c}{r} + \frac{1-\frac{c}{r}}{(1+r)^n}\right)$$

Terminology:

- $c = r$, then $P = F$, and "the bond sells at par"
- $c > r$, then $P > F$, and "the bond sells above par"
- $c < r$, then $P < F$, and "the bond sells $\underbrace{\text{at a discount}}_{\text{or "below par"}}$"

Note: Most simple type of bond: $\boxed{\text{zero coupon bond}}$, i.e., $c = 0$. (Then $P = \frac{F}{(1+r)^n}$.)