Stochastic Methods Lab, Jacobs University Bremen, Fall 2022

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Session 8
Sep. 29, 2022

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2. Options and Binomial Tree Models

2.1 Option Basics

Option: contract (or financial instrument) that depends on the future price of some other underlying asset (most commonly, stocks, which we will focus on)

=> this is called a "derivative" financial instrument

Option contract: Right to buy or sell underlying asset for "strike price" K at "expiration date" T.

Types of options:

- **Call option:** holder <u>can</u> <u>buy</u> underlying asset at price K at time T
- **Put option:** holder <u>can</u> <u>sell</u> underlying asset for price K at time T
- **European options:** can be exercised <u>only at</u> expiration date T
- **American options:** can be exercised <u>at or before</u> expiration T

Definitions: • price of underlying asset will be called $S(t)$

• payoff = value of the option at expiration time T

Ex.: strike price $K = 50\$$

↳ suppose at T, the stock price $S(T) = 60\$$

↳ call option (buy): payoff $= 60\$ - 50\$ = 10\$$ (exercise option)

↳ put option (sell): payoff $= 0\$$ (not exercise option)

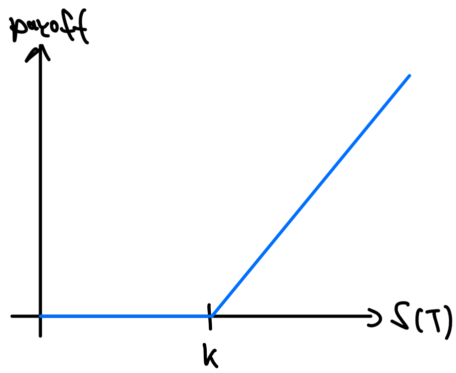
What are options good for?

- betting / speculation

- insurance

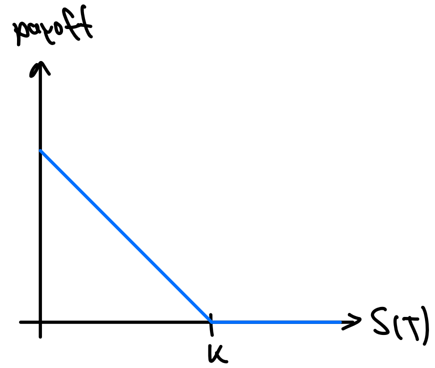
Payoffs:

call



$$\text{payoff } \underbrace{C}_{\text{call payoff}} = \max(0, S(T) - K)$$

put



$$\text{payoff } \underbrace{P}_{\text{put payoff}} = \max(0, K - S(T))$$

$$(0 \leq S(T) < \infty)$$

note: · buying option: "long position"
· selling option: "short position"

Goal for most of the rest of class:

What is a fair price of an option?

(Surprisingly, there is actually an answer, even though stock prices are not predictable.)

Assumptions:

- There is a risk-free market, which we take to be a bond market, with risk-free interest rate r, constant in time (e.g., US treasury bonds, or ECB bonds).
- Stocks and bonds can be bought and sold unlimitedly and without transaction costs.

Problem: stock price is uncertain
$\Rightarrow$ we need a probabilistic model for $S(t)$, $0 \leq t \leq T$

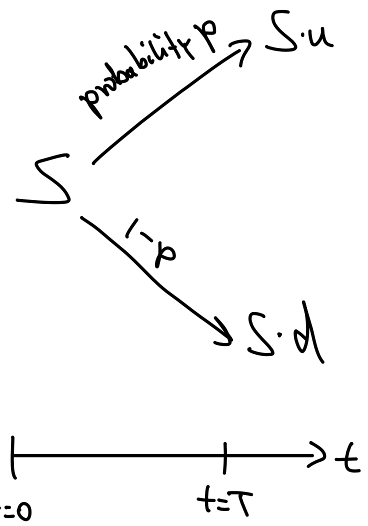
Main idea for "fair pricing":

no opportunity for risk-free profit

= no arbitrage assumption

2.2 Binary Model

First, simple model with only 2 possibilities and one time step:

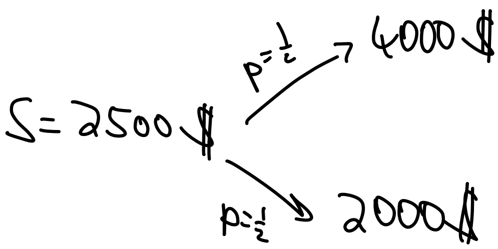


(p, u, d are the parameters of our model)

(d<u; keep in mind u>1, d<1)

Today, let's look at an example:

S = 2500 \$, K = 3000 \$, r = 0 , call



→ payoff = 1000 \$

→ payoff = 0 \$

Obvious idea: set price at $C = \frac{1}{2}(1000\,\$) + \frac{1}{2}\,0\,\$ = 500\,\$$

The option seller sells option at t=0 for 500 \$ and might have to sell a stock to the option buyer at t=T.

Possible strategies for option seller:

Strategy 1:

- At $t=0$: sell option $\Rightarrow$ profit 500 \$
- At $t=T$:
  - If $S(T) = 4000$ \$ (up-scenario), need to buy stock for 4000 \$ and sell it to option holder for 3000 \$

    $\Rightarrow$ profit $= 500\,\$ - 4000\,\$ + 3000\,\$ = -500\,\$$
  - If $S(T) = 2500$ (down-scenario) option is not exercised

    $\Rightarrow$ profit $= 500\,\$$

Strategy 2:

- At $t=0$: sell option (for 500 \$), borrow 2000 \$ and buy one stock (for 2500 \$)
- At $t=T$:
  - If $S(T) = 4000\,\$$ $\rightarrow$ option will be exercised, seller has to sell stock for $K = 3000\,\$$

    $\Rightarrow$ profit: $3000\,\$ - 2000\,\$ = 1000\,\$$

    (holder has made $1000\,\$ - 500\,\$ = 500\,\$$)
  - If $S(T) = 2000\,\$$ $\rightarrow$ option will not be exercised, one could sell stock for 2000 \$

    $\Rightarrow$ profit: $2000\,\$ - 2000\,\$ = 0\,\$$

    (holder has a balance of $-500\,\$$)

By following Strategy 2, option seller can always make a risk-free profit!

$\Rightarrow$ Price of 500 \$ was too high!

$\Rightarrow$ General idea: construct portfolio of stocks and bonds in such a way that the obligation can always be met, and which then mimics the option price, called "replicating portfolio".