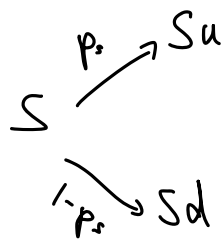
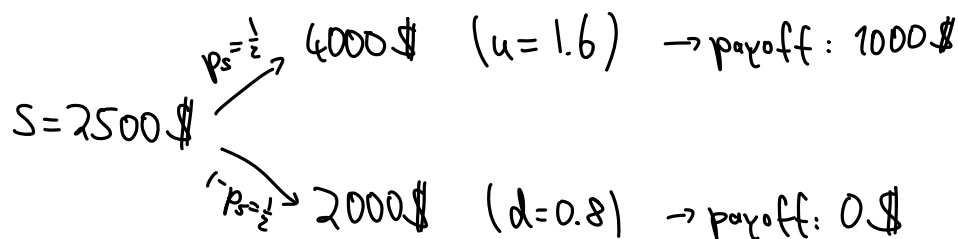


Prof. Dr. Soeren Petrat

(last time: binary model

Recall example:  $K = 3000 \$$  (call)

We dismissed the idea of option price = average w.r.t. stock market probabilities (from our model) because of the possibility of making risk-free profit.

New idea: Option price (value) = cost of a portfolio that leads to no risk-free profit for seller (meaning seller meets obligation exactly).

Such a portfolio is called **replicating portfolio**.

We define:

- $x_1$  = price of bond with risk-free interest rate  $r$
- $x_2$  = number of stocks at price  $S$  (also called "hedge ratio" or "delta")

We assume:

- continuous compounding of interest ( $FV = e^{rn} PV$ )
- $d < e^r < u$

$\Rightarrow$  cost of replicating portfolio is  $C = x_1 + Sx_2$

(value at expiration: • if stock goes up:  $e^r x_1 + Su x_2$   
• if stock goes down:  $e^r x_1 + Sd x_2$ )

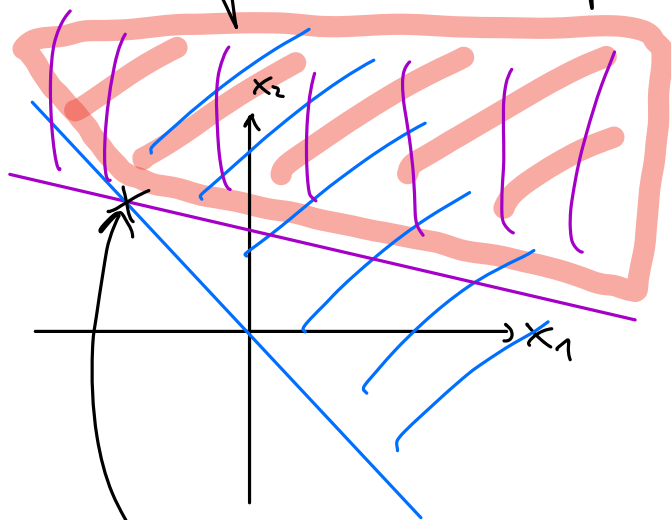
(Note: for simplicity we here assume  $e^r$  is the discount factor for the one period under consideration (one step).)

In general: riskless or zero profit for seller if

•  $e^r x_1 + Su x_2 \geq C_u$  , where  $C_u = \text{payoff in "up" scenario}$

•  $e^r x_1 + Sd x_2 \geq C_d$  , where  $C_d = \text{payoff in "down" scenario}$

(recall in general that for call options  $C_u = \max(0, Su - k)$ ,  $C_d = \max(0, Sd - k)$ .)



in this region seller would always make profit

no risk-free profit, obligation is exactly met; here we set up the replicating portfolio  $\Rightarrow$  option price

Option price is  $C = x_1 + Sx_2$  with  $x_1$  and  $x_2$  determined by:

- $e^r x_1 + S_u x_2 = C_u$

- $e^r x_1 + S_d x_2 = C_d$

Note: in our previous example we have

- $x_1 + 4000 x_2 = 1000$

- $x_1 + 2000 x_2 = 0$

$$\Rightarrow 2000 x_2 = 1000 \Rightarrow x_2 = \frac{1}{2}, x_1 = -1000$$

$$\Rightarrow \text{option price: } C = x_1 + Sx_2 = -1000\$ + 2500\$ \cdot \frac{1}{2} = 250\$$$

	seller: borrow 1000\$, get 250\$ for option, buy $\frac{1}{2}$ stock for 1250\$.	buyer: buy option for 250\$
$S(T) = 4000\$$ ("up" scenario)	<ul style="list-style-type: none"> <li>• need to buy another <math>\frac{1}{2}</math> stock for 2000\$</li> <li>• obligation: sell 1 stock for 3000\$ to buyer.</li> </ul> $\Rightarrow \text{profit: } -1000\$ - 2000\$ + 3000\$ = 0\$$	<ul style="list-style-type: none"> <li>• buy stock for 3000\$</li> </ul> $\Rightarrow \text{profit: } \underbrace{1000\$}_{4000\$ - 3000\$} - 250\$ = 750\$$
$S(T) = 2000\$$ ("down" scenario)	<ul style="list-style-type: none"> <li>• <math>\frac{1}{2}</math> stock is now worth 1000\$.</li> </ul> $\Rightarrow \text{profit: } -1000\$ + 1000\$ = 0\$$	$\Rightarrow \text{profit: } -250\$$

note: • one can buy  $\frac{1}{2}$  of a stock; this is called "fractional share"  
(also, e.g., for dividend reinvestments)

Generally, we need to solve

- $e^{-r}x_1 + Sux_2 = C_u$
- $e^{-r}x_1 + Sdx_2 = C_d$

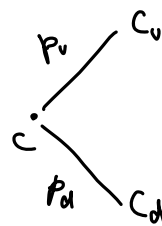
$$\Rightarrow \text{first eq. minus second: } Sux_2 - Sdx_2 = C_u - C_d \Rightarrow x_2 = \frac{1}{S} \frac{C_u - C_d}{u - d}$$

$$\begin{aligned} \text{To get } x_1: x_1 &= e^{-r}(C_d - Sdx_2) \\ &= e^{-r}\left(C_d - d \frac{C_u - C_d}{u - d}\right) \\ &= e^{-r}\left(\frac{C_d(u - d) - d(C_u - C_d)}{u - d}\right) \\ &= e^{-r}\left(\frac{uC_d - dC_u}{u - d}\right) \end{aligned}$$

option price is  $C = x_1 + Sx_2$

$$\begin{aligned} &= e^{-r}\left(\frac{uC_d - dC_u}{u - d}\right) + \frac{C_u - C_d}{u - d} \\ &= e^{-r}\left(\underbrace{\frac{u - e^{-r}}{u - d}}_{=: p_d} C_d + \underbrace{\frac{e^{-r} - d}{u - d}}_{=: p_u} C_u\right) \end{aligned}$$

$$\Rightarrow C = e^{-r}(p_d C_d + p_u C_u)$$



Note: •  $p_d + p_u = \frac{u - e^{-r} + e^{-r} - d}{u - d} = 1$

• since  $d < e^{-r} < u$ , we have that  $0 < p_d < 1$  and  $0 < p_u < 1$

$\Rightarrow$  it makes sense to call  $p_u, p_d$  probabilities, they are called **risk-neutral probabilities**.

Why? Look at expectation value of stock at time  $T$  under these risk-neutral

$$\text{probabilities: } E(S(T)_{p_u, p_d}) = p_u S_u + p_d S_d$$

$$= \frac{e^r - d}{u - d} S_u + \frac{u - e^r}{u - d} S_d$$

$$= S \left( \frac{ue^r - du + ud - e^r d}{u - d} \right)$$

$$= S e^r \Rightarrow \text{expected value is the same as for the risk-free bond market}$$

• remarkable: result  $C = e^{-r} (p_d C_d + p_u C_u)$  is independent of the probabilities from our stock model!

• In words: option price = discounted expectation value of the payoff under the risk-neutral probabilities