Stochastic Methods Lab, Jacobs University Bremen, Fall 2022

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Session 12
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Summary of last three sessions

Binary model for stock price:
$$S \begin{array}{l} \nearrow Su \\ \searrow Sd \end{array}$$

Binary model for option price:
$$C \begin{array}{l} \nearrow C_u = \text{payoff if stock goes up} \\ \searrow C_d = \text{payoff if stock goes down} \end{array}$$

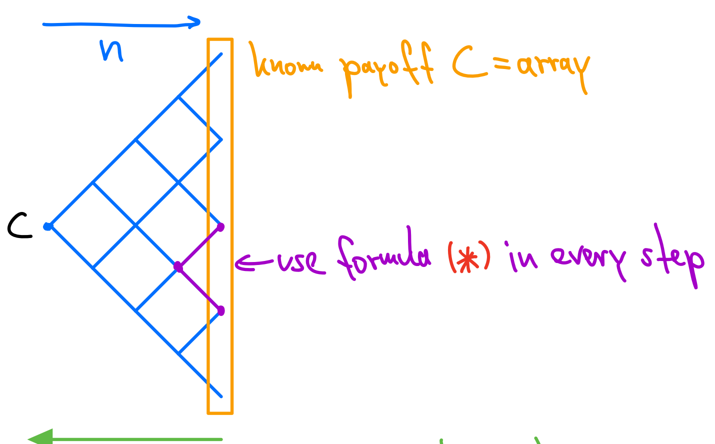
Option price = value of replicating portfolio $= C = \underbrace{x_1}_{\text{value of bonds}} + S \overbrace{x_2}^{\text{number of stocks}}$, with x_1, x_2 determined by

- $e^r x_1 + Su\, x_2 = C_u$
- $e^r x_1 + Sd\, x_2 = C_d$

Solving this gives $\underbrace{C = e^{-r}\left(p_d C_d + p_u C_u\right)}_{(*)}$ with $p_d = \dfrac{u-e^r}{u-d}$, $p_u = \dfrac{e^r - d}{u-d}$ $\quad (p_d + p_u = 1)$

"risk-neutral probabilities"

Repeating this yields a binomial tree model for option pricing:



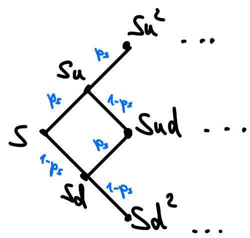
repeat until we have computed option price C
(backwards induction)

Note: binomial trees are very versatile and can be used, e.g., also for American options or with dividend payments. For the case of European call without dividends, an explicit formula is available (see also the discussion later today and in next session).

Question: How to choose parameters u and d?

For historical stock data, one usually considers expectation and variance of the rate of return $y(t)$ from $S(t) = S_0 e^{y(t)}$, i.e., $y(t) = \ln \frac{S(t)}{S_0}$. They are denoted by

$\mathbb{E}(y(t)) = \mu t$ and $\mathrm{Var}(y(t)) = \sigma^2 t$.

We want to match these to our tree model 

, where we find

$$\mathbb{E}(y(n \text{ steps})) = \left(\ln \frac{u}{d}\right) n p_s + n \ln d \quad \text{and} \quad \mathrm{Var}(y(n \text{ steps})) = \left(\ln \frac{u}{d}\right)^2 n p_s (1-p_s).$$

Now the choice $p_s = \frac{1}{2} + \frac{1}{2} \frac{\mu}{\sigma} \sqrt{\frac{t}{n}}$, $u = e^{\sigma \sqrt{\frac{t}{n}}}$, $d = \frac{1}{u}$ yields $\mathbb{E}(y(t))$ and $\mathrm{Var}(y(t))$ from above in the limit $n \to \infty$.

Noticable here is that u and d, and thus p_u and p_d, and thus the option price do <u>not</u> depend on the average growth of the stock μ, but only on its volatility σ.

2.5 Central Limit Theorem

For the limit $n \to \infty$ in our binomial tree model, we need to consider the limit $n \to \infty$ for the binomial distribution first:

$\nearrow$ "up" with probability p

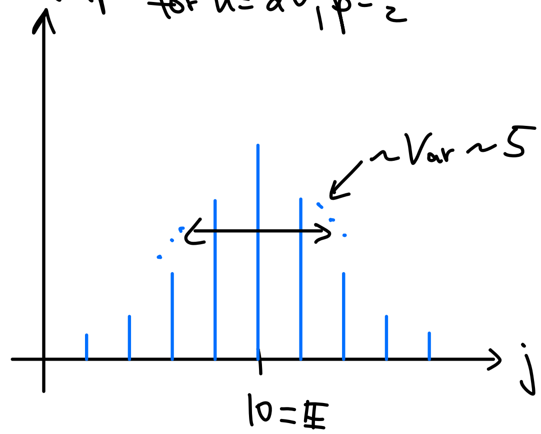
$\searrow$ "down" with probability $1-p$

Probability for j "up"s is $b(\underset{\text{number of "up"s}}{j};\overset{\text{total number of steps}}{n},\overset{\text{probability for "up"}}{p}) = \binom{n}{j} p^j (1-p)^{n-j}$ $\qquad \left(\binom{n}{j} = \frac{n!}{(n-j)!\,j!} \right)$

Recall: • $\mathbb{E}(j) = np$

• $\mathrm{Var}(j) = np(1-p)$

$b(j;n,p)$ for $n=20$, $p=\frac{1}{2}$



Note: in order to compare distributions (here, pictures for different n), we need to center and normalize the variance

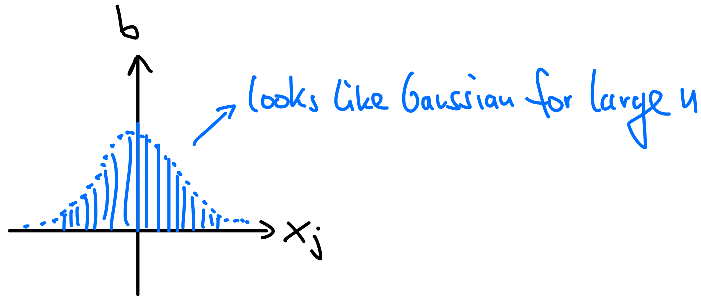
• centering: introduce $y_j = j - \mathbb{E}(j) = j - np$, such that

$$\mathbb{E}(y_j) = \mathbb{E}(j-np) = \mathbb{E}(j) - np = 0$$

• normalize variance (plus centering): $x_j = \dfrac{j-np}{\sqrt{np(1-p)}}$

$$\Rightarrow \mathrm{Var}(x_j) = \underset{\mathrm{Var}(\lambda X) = \lambda^2 \mathrm{Var}(X)}{\frac{1}{np(1-p)}} \overbrace{\mathrm{Var}(j-np)}^{=\mathrm{Var}(j) = np(1-p)} = 1$$

$\Rightarrow \quad j = \sqrt{np(1-p)}\, x_j + np$



Next: look at cumulative distribution, meaning the probability for A or fewer "up"s.

It is given by $\sum_{j=0}^{A} b(j,n,p)\, \underbrace{\Delta j}_{=1 = \text{distance between } j's}$

With the change of variables above, $j = \sqrt{np(1-p)}\, x_j + np$ and so $\Delta j = \sqrt{np(1-p)}\, \Delta x_j$,

so we should get (let A also depend on n)

$$\sum_{j=0}^{A_n} b(j,n,p)\, \Delta j = \sum_{x = \frac{-np}{\sqrt{np(1-p)}}}^{\frac{A_n - np}{\sqrt{np(1-p)}}} b\left(\sqrt{np(1-p)}\, x + np, n, p\right) \sqrt{np(1-p)}\, \Delta x$$

$\tilde{A} \leftarrow$ if A_n is chosen nicely (e.g., $A_n = np + \tilde{A}\sqrt{np(1-p)}$)

$$\text{should} \xrightarrow{n \to \infty} \int_{-\infty}^{\tilde{A}} \varphi(x)\, dx$$

some limiting fct.

Such a convergence result is called $\boxed{\text{Central Limit Theorem (CLT)}}$.

For the binomial distribution, we get:

$$\boxed{\begin{gathered} \sqrt{np(1-p)}\; b\left(\sqrt{np(1-p)}\, x + np, n, p\right) \xrightarrow{n \to \infty} \varphi(x) \quad \text{pointwise} \\ \text{with} \quad \varphi(x) = \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}}_{\text{normalized Gaussian}} = \mathcal{N}(0,1) \end{gathered}}$$

mean

variance

"normal distribution"

Remarks:

- Here we get pointwise convergence, but generally the CLT gives us convergence in the sense of cumulative distribution functions.

- Let's check normalization: $\left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx\right)^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; e^{-\left(\frac{x^2+y^2}{2}\right)}$

polar coordinates
$x^2+y^2=r^2$
$dxdy = r dr d\varphi$

$$= \frac{1}{2\pi} \int_0^{\infty} dr \int_0^{2\pi} d\varphi \; r e^{-\frac{r^2}{2}}$$

$$= \int_0^{\infty} dr \; r e^{-\frac{r^2}{2}}$$

$$= -e^{-\frac{r^2}{2}} \Big|_0^{\infty}$$

$$= 1$$

- One can also check that indeed $\mathbb{E}(x) = 0$ and $\mathrm{Var}(x) = 1$

Ingredients for the proof:

- We need to approximate factorials with the $\boxed{\text{Stirling approximation } n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n}$

Motivation why this is true: $\ln n! = \sum_{i=1}^{n} \ln(i) \approx \int_1^n \ln(x) dx = \int_1^n 1 \cdot \ln(x) dx$

$$= x \ln x \Big|_1^n - \int_1^n x \frac{1}{x} dx = n \ln(n) - (n-1) \approx n\ln(n) - n$$

$\ln a^b = b \ln a$

$$\Rightarrow n! = e^{\ln n!} \approx e^{n\ln(n) - n} = e^{\ln n^n} e^{-n} = n^n e^{-n} = \left(\frac{n}{e}\right)^n$$

(the factor $\sqrt{2\pi n}$ (or even higher order terms) can be found with more careful arguments)

- Taylor expansion