2.6 Black-Scholes Formula

Recall the exact formula for the price of European call options:

$$C = e^{-rT} \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \max(0, Sv^j d^{n-j} - K) \quad (K \text{ strike price, } r \text{ period interest rate})$$

$$= e^{-rT} \mathbb{E}(\text{payoff})$$

$\hookrightarrow$ under binomial distribution

Many terms in the sum are 0; payoff $\neq 0$ if $Sv^j d^{n-j} - K > 0$

$$\Rightarrow \text{need} \quad v^j d^{n-j} > \frac{K}{S} \Rightarrow \left(\frac{v}{d}\right)^j > \frac{K}{Sd^n} \Rightarrow j > \frac{\ln\left(\frac{K}{Sd^n}\right)}{\ln\left(\frac{v}{d}\right)} = \frac{\ln\left(\frac{K}{S}\right) - n \ln d}{\ln\left(\frac{v}{d}\right)} =: A_n$$

$$\Rightarrow C = e^{-rT} \sum_{j=A_n}^{n} \binom{n}{j} p^j (1-p)^{n-j} \left(Sv^j d^{n-j} - K\right)$$

$$= S \sum_{j=A_n}^{n} \binom{n}{j} \left(pve^{-r\frac{T}{n}}\right)^j \left((1-p)d\, e^{-r\frac{T}{n}}\right)^{n-j} - Ke^{-rT} \sum_{j=A_n}^{n} \binom{n}{j} p^j (1-p)^{n-j}$$

Recall $p = \dfrac{e^{r\frac{T}{n}} - d}{v - d} \Rightarrow 1-p = \dfrac{v - d - e^{r\frac{T}{n}} + d}{v - d} = \dfrac{v - e^{r\frac{T}{n}}}{v - d}$

$$\Rightarrow (1-p)d\, e^{-r\frac{T}{n}} = \frac{v - e^{r\frac{T}{n}}}{v - d}\, d e^{-r\frac{T}{n}} = \frac{vde^{-r\frac{T}{n}} - d + v - v}{v - d} = 1 - pve^{-r\frac{T}{n}}$$

$$\Rightarrow C = S \sum_{j=A_n}^{n} b\left(j, n, pve^{-r\frac{T}{n}}\right) - Ke^{-r\frac{T}{n}} \sum_{j=A_n}^{n} b(j, n, p)$$

Next: use our calibration $v = e^{6\sqrt{\frac{T}{n}}}$, $d = \frac{1}{v} = e^{-6\sqrt{\frac{T}{n}}}$

$$\Rightarrow A_n = \frac{\ln(\frac{K}{S}) - n \ln d}{\ln(\frac{v}{d})} = \frac{\ln(\frac{K}{S}) - n \ln e^{-6\sqrt{\frac{T}{n}}}}{\ln e^{26\sqrt{\frac{T}{n}}}} = \frac{\ln(\frac{K}{S}) + 6\sqrt{T}\sqrt{n}}{26\sqrt{\frac{T}{n}}} = \frac{\ln(\frac{K}{S})}{26\sqrt{T}}\sqrt{n} + \frac{1}{2}n$$

$$\Rightarrow p = \frac{e^{r\frac{T}{n}} - d}{v - d} = \frac{e^{r\frac{T}{n}} - e^{-6\sqrt{\frac{T}{n}}}}{e^{6\sqrt{\frac{T}{n}}} - e^{-6\sqrt{\frac{T}{n}}}} \overset{\text{Taylor}}{\underset{\downarrow}{=}} \frac{1 + r\frac{T}{n} + O(n^{-2}) - \left[1 - 6\sqrt{\frac{T}{n}} + \frac{1}{2}6^2\frac{T}{n} + O(n^{-\frac{3}{2}})\right]}{1 + 6\sqrt{\frac{T}{n}} + \frac{1}{2}6^2\frac{T}{n} + O(n^{-\frac{3}{2}}) - \left[1 - 6\sqrt{\frac{T}{n}} + \frac{1}{2}6^2\frac{T}{n} + O(n^{-\frac{3}{2}})\right]}$$

"big O" notation

$$= \frac{6\sqrt{\frac{T}{n}} + (r - \frac{6^2}{2})\frac{T}{n} + O(n^{-\frac{3}{2}})}{26\sqrt{\frac{T}{n}} + O(n^{-\frac{3}{2}})}$$

$$= \frac{1}{2}\left(1 + \frac{(r - \frac{6^2}{2})}{6}\sqrt{\frac{T}{n}} + O(n^{-1})\right)$$

$$= \frac{O(n^{-\frac{3}{2}})}{26\sqrt{T}n^{-\frac{1}{2}}}$$

Then the integral boundary from the CLT is

$$\hat{A} = \lim_{n \to \infty} \frac{A_n - np}{\sqrt{np(1-p)}} = \lim_{n \to \infty} \frac{\frac{\ln(\frac{K}{S})}{26\sqrt{T}}\sqrt{n} + \frac{1}{2}n - \left[\frac{n}{2} + \frac{n}{2}\frac{(r - \frac{6^2}{2})}{6}\sqrt{\frac{T}{n}} + O(n^{-1})\right]}{\sqrt{n}\left(\frac{1}{2} + \frac{\text{const}}{\sqrt{n}} + O(n^{-1})\right)^{\frac{1}{2}}\left(\frac{1}{2} - \frac{\text{const}}{\sqrt{n}} + O(n^{-1})\right)^{\frac{1}{2}}}$$

$$= \sqrt{n}\sqrt{\frac{1}{4} + O(n^{-1})} = \sqrt{n}\frac{1}{2} + O(n^{-\frac{1}{2}})$$

$$= \frac{\ln(\frac{K}{S}) - (r - \frac{6^2}{2})T}{6\sqrt{T}}$$

Now applying the CLT gives us: (after some more computation)

$$C = S\,\Phi(x) - Ke^{-rT}\,\Phi(x - \sigma\sqrt{T})$$, the Black-Scholes formula,

with cumulative normal distribution $\Phi(x) = \int_{-\infty}^{x} \varphi(y)\,dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}}\,dy$

↓
normal gaussian

and $x = \dfrac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$

let's check: $\hat{A} = \dfrac{\ln\left(\frac{K}{S}\right) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = -\left(\dfrac{\ln\left(\frac{S}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \right)$

$$= -\left(x - \sigma\sqrt{T}\right)$$

Then: $Ke^{-rT} \sum\limits_{j=A_n}^{n} b(j; n, p) \xrightarrow[CLT]{n \to \infty} Ke^{-rT} \int\limits_{\hat{A}}^{\infty} \varphi(y)\,dy = Ke^{-rT} \int\limits_{-(x-\sigma\sqrt{T})}^{\infty} \varphi(y)\,dy$

$$= Ke^{-rT} \int\limits_{-\infty}^{x - \sigma\sqrt{T}} \varphi(y)\,dy$$

$$= Ke^{-rT}\,\Phi(x - \sigma\sqrt{T})$$

Similar for the first summand, but we need to use approximations for $pue^{-r\frac{T}{n}}$.

2.7 Convergence Rates

Consider some sequence $C_n \xrightarrow{n \to \infty} C$, e.g., $C_n =$ option price for n step binomial tree, and

$C =$ price from Black-Scholes

It is very important to know how fast the convergence is.

Often, the convergence goes like a power law: $E_n = |C_n - C| \approx A n^{-\beta}$ for large n
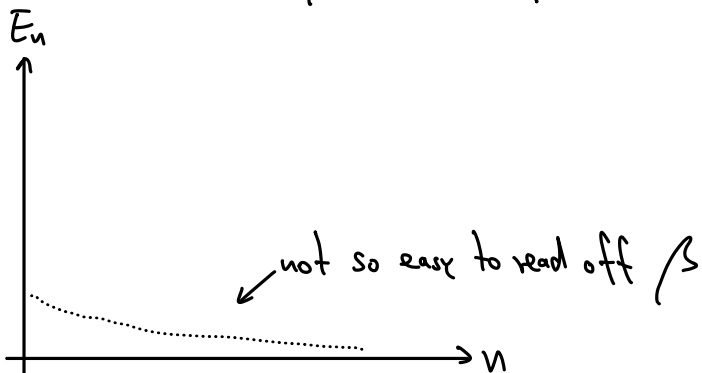
β is called rate of convergence

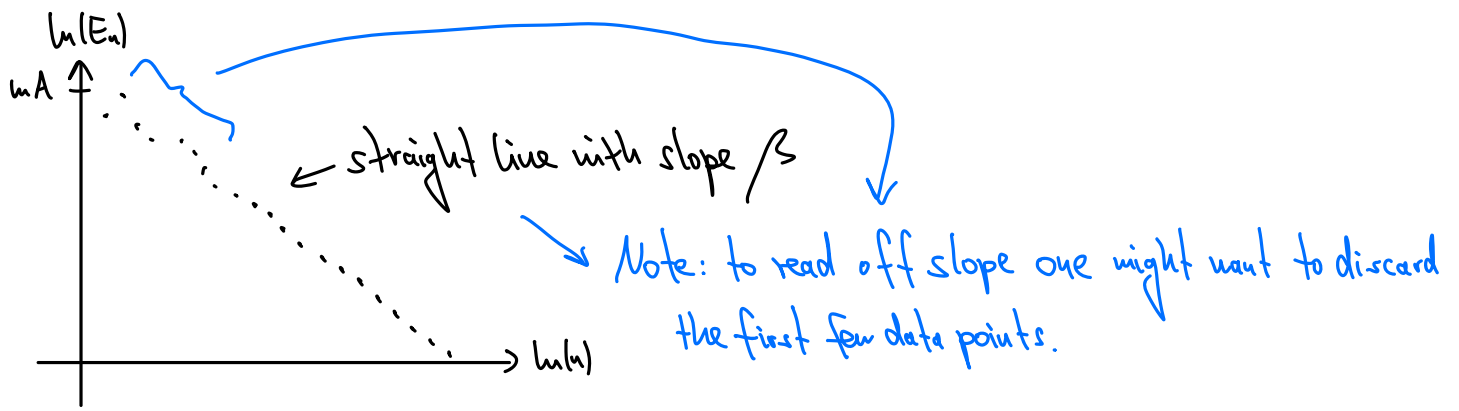Remarks: • different β can make a huge difference (e.g. linear vs. quadratic)

• if limit C is unknown one could, e.g., look at $N \gg n$ and then

$|C_n - C_N|$, or $E_n = |C_{n+1} - C_n|$

How can we read off β from a plot?

not so easy to read off β

better: $\ln E_n = \ln A n^{-\beta} = \ln A - \beta \ln n$

$\ln(E_n)$

mA

← straight line with slope β

Note: to read off slope one might want to discard the first few data points.

$\ln(n)$

python: $\log\log(n, E_n)$ produces such plots