## 3.3 Stochastic Differential Equations

Usual first-order ordinary differential equation (ODE):

$$\frac{dX(t)}{dt} = f(X(t),t)$$

We could also write this in integral form: $X(t) = X(0) + \int_0^t f(X(s),s)\,ds$

A stochastic differential equation (SDE) can be written down in integral form:

$$\boxed{X(t) = X(0) + \int_0^t f(X(s),s)\,ds + \underbrace{\int_0^t g(X(s),s)\,\underbrace{dW(s)}_{\text{Brownian motion increments}}}_{\text{stochastic integral, Itô from now on}}}$$

As a short-hand notation, we often write: $dX(t) = f(X(t),t)\,dt + g(X(t),t)\,dW(t)$

Today: Examples, numerical solutions and their error; next time: how to find solutions

Ex.: Next time, we will see that GBM $S(t) = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right)$

satisfies the SDE $dS(t) = \mu S(t)\,dt + \sigma S(t)\,dW(t)$

In integral form: $\boxed{S(t) - S_0 = \mu \int_0^t S(u)\,du + \sigma \int_0^t S(u)\,dW(u)}$

Even without knowing the solution, we can figure out its expectation value:

$$\mathbb{E}(S(t)) - S_0 = \mu \int_0^t \mathbb{E}(S(u))\,du + \sigma \int_0^t \underbrace{\mathbb{E}(S(u)\,dW(u))}$$

$$= \mathbb{E}(S(u))\,\underbrace{\mathbb{E}(dW(u))}_{=0} = 0$$

b.c. $dW(u)$ and $S(u)$ are independent (Itô integral!)

$$\Rightarrow \mathbb{E}(S(t)) - S_0 = \mu \int_0^t \mathbb{E}(S(u))\,du \qquad \left( \frac{d\mathbb{E}(S(t))}{dt} = \mu\, \mathbb{E}(S(t)) \right)$$

$$\Rightarrow \text{solution: } \mathbb{E}(S(t)) = S_0 e^{\mu t}$$

Next: numerical solutions

Usual ODEs $\frac{dX(t)}{dt} = f(X(t), t)$ can be solved numerically with $\boxed{\text{Euler's method:}}$

partition $[0,t]$ into $N$ steps, $x_n = n\Delta t$, $\Delta t = \frac{T}{N}$ and then write down discretized eq.:

$$\boxed{\frac{x_{n+1} - x_n}{\Delta t} = f(x_n, t_n) \quad \Rightarrow \quad x_{n+1} = x_n + f(x_n, t_n)\Delta t}$$

What is the error?

In one step: • Euler: $\overset{x(\Delta t)}{\overset{\shortparallel}{x_1}} = x_0 + f(x_0, \overset{t_0}{\overset{\shortparallel}{0}})\,\Delta t$

• Taylor expansion of exact solution $X(\Delta t) = x(0) + \overbrace{x'(0)}^{= f(x(0),0)}\Delta t + \frac{1}{2}x''(0)(\Delta t)^2 + \mathcal{O}(\Delta t^3)$

$$\Rightarrow |x(\Delta t) - x_1| \approx \text{const}\,(\Delta t)^2 \qquad \left(\Delta t = \frac{T}{N}\right)$$

$$\Rightarrow \text{total error } |X(t) - x_N| \approx \sum_{i=1}^{N} \text{const}(\Delta t)^2 \approx \frac{c(t)}{N}$$

The same works for SDEs; it is then called $\boxed{\text{Euler-Maruyama method}}$:

$$\boxed{X_{n+1} = X_n + f(X_n, t_n)\Delta t + g(X_n, t_n)\Delta W_n}$$

For the error, there are two often used definitions:

- $\boxed{\text{strong error: } \mathbb{E}(|X(t) - X_N|) \approx c_s (\Delta t)^{\alpha}}$, $\alpha$ = strong order of convergence

The relevance for individual paths comes from Markov's inequality:

$$\mathbb{P}(|X| > a) \leq \frac{\mathbb{E}(|X|)}{a} \qquad (a > 0)$$

Quick proof: $\mathbb{E}(|X|) = \int_{-\infty}^{\infty} |x| \underbrace{\rho(x)}_{\text{probability density}} dx$

$$= \left( \int_{-\infty}^{-a} + \int_{-a}^{a} + \int_{a}^{\infty} \right) |x| \rho(x) dx$$

$$\geq \left( \int_{-\infty}^{-a} + \int_{a}^{\infty} \right) |x| \rho(x) dx$$

$$\geq a \underbrace{\left( \int_{-\infty}^{-a} + \int_{a}^{\infty} \right) \rho(x) dx}_{\mathbb{P}(|x| > a)}$$

So applying Markov to the strong error, we, e.g., get

$$\mathbb{P}\left(|X(t) - X_N| > (\Delta t)^{\frac{\alpha}{2}}\right) \leq \frac{\mathbb{E}(|X(t) - X_N|)}{(\Delta t)^{\frac{\alpha}{2}}} \approx c_s \frac{(\Delta t)^{\alpha}}{(\Delta t)^{\frac{\alpha}{2}}} = c_s (\Delta t)^{\frac{\alpha}{2}}$$

=> Strong error also tells us about error for individual paths

- $\boxed{\text{weak error:}}$ $|\mathbb{E}(X(t)) - \mathbb{E}(X_N)| \approx C_w (\Delta t)^{\beta}$, $\beta$ = weak order of convergence

note: $|\mathbb{E}(X(t) - X_N)| \leq \mathbb{E}(|X(t) - X_N|)$, so weak error $\leq$ strong error

$\left( |\int f(x) dx| \leq \int |f(x)| dx \right)$

But weak error does not necessarily tell us something about individual paths.

For example, compare $W(t)$ with $0$ $\Rightarrow$ $\mathbb{E}(W(t)) - \mathbb{E}(0) = 0 - 0 = 0$,
but $W(t)$ is very different from $0$.