3.4 Itô-lemma

Goal: find a stochastic version of the chain rule

Non-stochastic: given $h(x,t)$, we have $dh = \frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial t} dt$

Now: $h(W(t),t) = h(x,t)\big|_{x=W(t)}$

Let us write the integral form for the simpler case $h(W(t))$ (no explicit t-dependence):

$$h(W(T)) - h(W(0)) = \sum_{j=0}^{n-1} \Big[\underbrace{h(W(t_{j+1}))} - h(W(t_j)) \Big]$$

Taylor expansion around $W(t_j)$: $h(W_{t_{j+1}}) = h(W(t_j)) + h'(W(t_j))\big(W(t_{j+1}) - W(t_j)\big)$

$$+ \frac{1}{2} h''(W(t_j))\big(W(t_{j+1}) - W(t_j)\big)^2 + \mathcal{O}(\Delta W_j^3)$$

$$\Rightarrow h(W(T)) - h(W(0)) = \sum_{j=0}^{n-1} \Big(h'(W(t_j))\, \Delta W_j + \frac{1}{2} h''(W(t_j))(\Delta W_j)^2 + \mathcal{O}(\Delta W_j^3) \Big)$$

$$\xrightarrow{n\to\infty} \int_0^T h'(W)\, dW + \int_0^T \frac{1}{2} h''(W(t))\, dt$$

remember from last time $(\Delta W_j)^2 \sim \underbrace{\Delta t}_{\text{deterministic}} + \underbrace{\text{Rest}}\ \mathcal{O}(\Delta t^2)$

stochastic, but vanishes in the limit: $\sum \Delta t^2 \to 0$

$$h(W(T)) - h(W(0)) = \int_0^T \underbrace{\left(\frac{\partial h}{\partial x}\right)(W(t))}_{\text{first derivative, evaluated at } x = W(t)} dW(t) + \int_0^T \frac{1}{2}\left(\frac{\partial^2 h}{\partial x^2}\right)(W(t))\, dt$$

In short-hand notation: $dh = h' dW + \frac{1}{2} h'' dt$

Now if there is also explicit t-dependence, i.e., $h = h(W(t), t)$, then the following Itô formula holds:

$$\boxed{dh = h' dW + \left(\dot{h} + \frac{1}{2} h''\right) dt} \qquad \left(\text{with } \dot{h}(W(t), t) = \frac{\partial h(x,t)}{\partial t}\Big|_{x = W(t)}\right)$$

$$\left(dh = \left(\frac{\partial h}{\partial x}\right)(W(t))\, dW(t) + \left(\frac{\partial h}{\partial t} + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}\right)(W(t))\, dt \right)$$

Let's consider a few examples:

- $h(W(t), t) = W(t)^2$

  Apply Itô formula: $dh = 2W(t)\, dW(t) + \frac{1}{2} 2\, dt = 2W dW + dt$

  is the SDE (with solution $h = W^2$)

  With that, we can, e.g., compute the expectation value:

$$\mathbb{E}(W(t)^2) - \mathbb{E}(\underbrace{W(0)^2}_{=0}) = 2\int_0^t \underbrace{\mathbb{E}(W(s)\, dW(s))}_{= \mathbb{E}(W(s))\underbrace{\mathbb{E}(dW(s))}_{=0}} + \mathbb{E}\Big(\underbrace{\int_0^t ds}_{=t}\Big)$$

independence, like we saw last time

$$\Rightarrow \mathbb{E}(W(t)^2) = t$$

- similar example: $h = W^4$

$$\Rightarrow dh = 4W^3 dW + \frac{1}{2} 4 \cdot 3 W^2 dt = 4W^3 dW + 6W^2 dt$$

$$\Rightarrow \mathbb{E}(W(T)^4) = \mathbb{E}\left(6 \int_0^T W(t)^2 dt\right) = 6 \int_0^T \underbrace{\mathbb{E}(W(t)^2)}_{= t\text{, as we computed above}} dt$$

$$= 6 \int_0^T t\, dt = 3T^2$$

similarly one could compute $\mathbb{E}(W^{2n})$

- The Itô formula also allows us to solve SDEs. As example, consider:

$$dh = h^3 dt - h^2 dW \quad , h(0) = 1$$

Itô formula: $dh = h' dW + (\dot{h} + \frac{1}{2} h'') dt$

We have now reduced an SDE to $\underbrace{\text{two PDEs}}_{\text{those have no stochasticity any more}}$!

$$\Rightarrow \underbrace{h' = -h^2}\quad \text{and} \quad \dot{h} + \frac{1}{2} h'' = h^3 \quad \text{by comparison}$$

$$\frac{dh}{dx} = -h^2 \overset{\text{seperation of variables}}{\Rightarrow} \frac{dh}{-h^2} = dx \overset{\text{integrate}}{\Rightarrow} \underbrace{\int_0^x \frac{dh}{-h^2}} = \underbrace{\int dx}_{= x}$$

$$= h^{-1} \Big|_0^x$$

$$= \frac{1}{h(x)} - \underbrace{\frac{1}{h(0)}}_{=1} = \frac{1}{h(x)} - 1$$

$$\Rightarrow \frac{1}{h(x)} = x + 1 \Rightarrow h(x) = \frac{1}{x+1}$$

$$\Rightarrow \frac{1}{2} h'' = \frac{1}{2}\left((x+1)^{-1}\right)'' = \frac{1}{2}(-1)(-2)(x+1)^{-3} = (x+1)^{-3} = h(x)^3, \text{ so solution}$$

has no explicit time-dependence and is compatible with second eq. above

$$\Rightarrow h(W(t)) = \frac{1}{W(t)+1} \qquad (\text{has singularities for finite } t)$$

Hints for HW7:

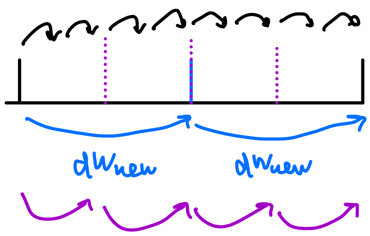
Problem 1: • S_N = Euler-Maruyama after N steps (corresponding to T, i.e., $\Delta t = \frac{T}{N}$)

• By def., Euler-Maruyama is inductive, i.e., it has to be implemented with a "for" loop. In the special case of GBM, one could also use cumprod.

• For strong/weak error: $\mathbb{E}$ is over ensemble, N is varied (# of steps in Euler-Maruyama)

• Note: weak error might be a bit hard to read off; try to fit a line by hand anyway.

• For the error rate, one could start with $N_{max} = 2{*}{*}k$ $(k \simeq 10)$



$N = (2, 2^2, 2^3, \dots, \underbrace{2{*}{*}k}_{N_{max}})$

For each realization, create dW of length N_{max}

↳ with that, compute GBM

↳ compute $S_{N_{max}}$ (with the dW above)

↳ compute $S_{\frac{N_{max}}{2}}$ by using Euler-Maruyama with coarsened new

$dW_{new_{\frac{N_{max}}{2}}} = (dW_0 + dW_1, dW_1 + dW_2, \dots)$

↳ repeat till $\frac{N_{max}}{2{*}{*}k}$ or sth. smaller

Problem 4: Imagine two scenarios:

a) You want to keep the stock till after expiration. Is it then better to exercise early, e.g., when $S_t > K$, or at expiration?

b) You want to make profit immediately by exercising the option early, when $S_t > K$, and selling the stock. Is it better to exercise option, or to sell the option?