4. Black-Scholes Equation and Finite Difference Schemes

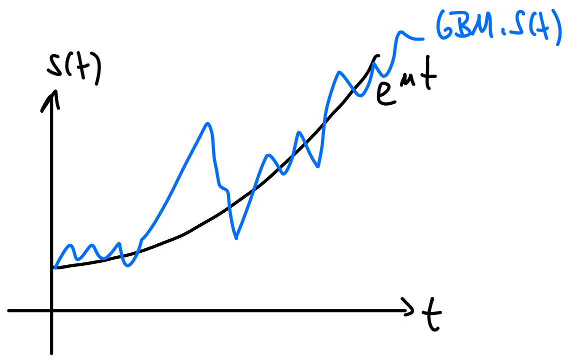
4.1 Derivation of the Black-Scholes Equation

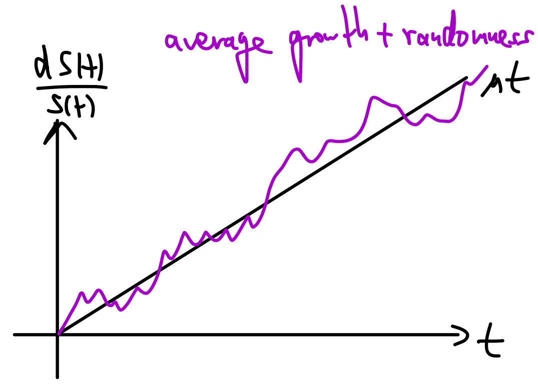
We assume that the stochastic process for stock prize development is geometric Brownian motion (GBM):

$$dS = \mu S dt + \sigma S dW$$

This means: Stock's rate of return $\frac{dS}{S} = \mu dt + \sigma dW$ has expectation μdt and variance $\sigma^2 dt$.

To visualize:





"Stocks behave like regular cash-flows/bonds ($\frac{dX}{X} = r dt$), but with risk (σdW term)."

Recall: From Itô's lemma we found that $S(t) = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)}$.

Now: Option price C is a function of $S(t)$ and t, so $C = C(S(t), t) = C(x,t)\big|_{x=S(t)}$

Recall Itô's lemma: If $X(t)$ is sol. to $dX = f dt + g dW$, and $F(X(t), t)$, then

$$dF = \left[\frac{\partial F}{\partial t} + f \frac{\partial F}{\partial x} + \frac{1}{2} g^2 \frac{\partial^2 F}{\partial x^2} \right] dt + g \frac{\partial F}{\partial x} dW$$

So in our case ($f = \mu S, g = \sigma S$):

$$dC = \left[\frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right] dt + \sigma S \frac{\partial C}{\partial S} dW$$

Merton's trick: consider a portfolio of value Π that eliminates risk

$$\Rightarrow \quad \underbrace{\Pi}_{\text{bonds}} = \alpha \underbrace{C}_{\text{option}} + \beta \underbrace{S}_{\text{stock}} \qquad \text{(replicating portfolio)}$$

$$d\Pi = \alpha \, dC + \beta \, dS$$

$$= \alpha \left[\frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right] dt + \alpha \sigma S \frac{\partial C}{\partial S} dW + \beta \mu S dt + \beta \sigma S dW$$

To eliminate risk, we need $\beta = -\alpha \frac{\partial C}{\partial S}$.

With that choice, we have

$$d\Pi = \alpha \left[\frac{\partial C}{\partial t} + \cancel{\mu S \frac{\partial C}{\partial S}} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - \cancel{\mu S \frac{\partial C}{\partial S}} \right] dt$$

Now there is no randomness in $d\Pi$ anymore, so Π has to grow with riskless rate r:

$$\Pi(t) = \Pi(0) e^{rt}, \text{ or } d\Pi = \Pi r \, dt = \alpha \left(C - \frac{\partial C}{\partial S} S \right) r \, dt$$

(Otherwise there would be the possibility of risk-free profit.)

Setting both expression for $d\Pi$ equal yields

$$\Rightarrow \boxed{\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} = rC}$$

This is the $\boxed{\text{Black-Scholes(-Merton) equation.}}$

Remarks:

- This is a partial differential equation (PDE), first order in time, second order in S
- We know the "initial condition" $C(S, t=T) = \text{payoff}$, e.g. for European calls, we have

  $C(S,T) = \max(S-K, 0)$, K = strike price, T = expiration.

  We want to solve for $C(S, t=0)$

  ↳ Black-Scholes eq. is a backward drift-diffusion equation

  ($\frac{\partial^2 C}{\partial S^2}$ is a "diffusion" term, $\frac{\partial C}{\partial S}$ a "drift" term)

- We have the boundary condition $C(S=0, t) = 0$ for all $t \in [0, T]$

- By a change of variables, the eq. can be transformed into

  $$\frac{\partial \theta}{\partial u} = \frac{1}{2}\frac{\partial^2 \theta}{\partial z^2} \quad (\theta = \theta(z,u))$$, the heat equation

- Option price $C(S,0)$ at time $t=0$ depends on the parameters r, σ, K, T, but not on μ. (Analogous to bin. tree model, where option price is independent of stock market probabilities.)