4.2 Connection between Black-Scholes Equation and Formula

B.-S. eq.: $\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC$

European calls: "initial" condition $C(S,T) = \max(S-K,0)$

Generally: boundary condition $C(0,t) = 0$

One could do several changes of variables to reduce B.-S. eq. to the heat eq.

Ex.: One can remove the rC term by the following change of variables:

$$C(S,t) = B(S,\tau)e^{-r\tau}K \quad \text{with } \tau = T-t$$

Then: $\bullet\ \frac{\partial C}{\partial t} = \frac{\partial C}{\partial \tau}\underbrace{\frac{\partial \tau}{\partial t}}_{=-1} = -\left(\frac{\partial B}{\partial \tau}e^{-r\tau}K + \underbrace{B(-r)e^{-r\tau}K}_{-rC}\right)$

$$= -\frac{\partial B}{\partial \tau}e^{-r\tau}K + rC$$

$\bullet\ \frac{\partial C}{\partial S} = \frac{\partial B}{\partial S}e^{-r\tau}K$

$\Rightarrow$ B.-S. eq. becomes: $-\frac{\partial B}{\partial \tau}e^{-r\tau}K + \underline{rC} + rS\frac{\partial B}{\partial S}e^{-r\tau}K + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 B}{\partial S^2}e^{-r\tau}K = \underline{rC}$

$\Rightarrow\ -\frac{\partial B}{\partial \tau} + rS\frac{\partial B}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 B}{\partial S^2} = 0$

with initial condition: $C(S,T) = B(S,0)K \overset{!}{=} \max(S-K,0)$

$\Rightarrow \mathcal{B}(S,\tau=0) = \max\left(\frac{S}{K}-1, 0\right)$

and boundary condition $\mathcal{B}(0,\tau) = 0$

With similar changes of variables one can remove the prefactors and $\frac{\partial C}{\partial S}$ term, and thus reduce the B.-S. eq. to the heat eq.:

$$\underbrace{\frac{\partial \Theta}{\partial u} = \frac{1}{2}\frac{\partial^2 \Theta}{\partial z^2}}_{\text{the highest derivatives always remain}} \quad \text{with initial condition } \Theta(z,0) = \max(1-e^{-z}, 0)$$

$t \to \tau \to u$ $\quad$ $S \to z$

(z as a fct. of S also depends on K)

note: K is hidden in the change of variables from S to z.

The heat eq. can be solved with Fourier transform:

$$\hat{\Theta}(p,u) = \frac{1}{\sqrt{2\pi}} \int e^{ipz}\, \Theta(z,u)\, dz$$

$$\Theta(z,u) = \frac{1}{\sqrt{2\pi}} \int e^{-ipz}\, \hat{\Theta}(p,u)\, dp$$

keeping u fixed

$$\Rightarrow \text{heat eq. becomes } \underbrace{\frac{1}{\sqrt{2\pi}} \int e^{-ipz} \frac{\partial \hat{\Theta}(p,u)}{\partial u}\, dp}_{\frac{\partial \Theta}{\partial u}} = \underbrace{\frac{1}{\sqrt{2\pi}} \int \frac{1}{2}(-ip)^2 e^{-ipz}\, \hat{\Theta}(p,u)\, dp}_{\frac{1}{2}\frac{\partial^2 \Theta}{\partial z^2}}$$

$$\Rightarrow \text{need to solve } \frac{\partial \hat{\Theta}(p,u)}{\partial u} = \frac{1}{2}(-ip)^2\, \hat{\Theta}(p,u) = -\frac{1}{2}p^2\, \hat{\Theta}(p,u)$$

solution: $\hat{\Theta}(p,u) = e^{-\frac{1}{2}p^2 u}\, \hat{\Theta}(p,0)$

$$\Rightarrow \Theta(z,u) = \frac{1}{\sqrt{2\pi}} \int e^{-ipz}\, \hat{\Theta}(p,u)\, dp$$

$$= \frac{1}{\sqrt{2\pi}} \int e^{-ipz} e^{-\frac{1}{2}p^2 u}\, \underbrace{\hat{\Theta}(p,0)}\, dp$$

$$= \frac{1}{\sqrt{2\pi}} \int e^{ipy}\, \Theta(y,0)\, dy$$

$$= \frac{1}{2\pi} \int \underbrace{\left[\int e^{-ipz} e^{ipy} e^{-\frac{1}{2}p^2 u}\, dp \right]}_{\text{fct. of } y \text{ (and } z,u)} \Theta(y,0)\, dy$$

Now: $\int e^{-ip(z-y)} e^{-\frac{1}{2}up^2}\, dp = \int e^{-\frac{1}{2}up^2 - ip(z-y)}\, dp$

$$= \int e^{-\frac{u}{2}\left[p^2 + 2p\frac{i(z-y)}{u} + \left(\frac{i(z-y)}{u}\right)^2 - \left(\frac{i(z-y)}{u}\right)^2\right]}\, dp$$

$$= \int e^{-\frac{u}{2}\left[p + \frac{i(z-y)}{u}\right]^2} dp\; e^{-\frac{u}{2}\left(-\left(\frac{i(z-y)}{u}\right)^2\right)}$$

$p + \frac{i(z-y)}{u} = \tilde{p}$

$\Rightarrow dp = d\tilde{p}$

$$= \int e^{-\frac{u}{2}\tilde{p}^2}\, d\tilde{p}\; e^{-\frac{(z-y)^2}{2u}}$$

$\tilde{p} = \frac{k}{\sqrt{u}}$

$$= \frac{1}{\sqrt{u}} \underbrace{\int e^{-\frac{k^2}{2}}\, dk}_{=\sqrt{2\pi}}\; e^{-\frac{(z-y)^2}{2u}}$$

$$= \sqrt{\frac{2\pi}{u}}\, e^{-\frac{(z-y)^2}{2u}}$$

convolution

// $(G_u * \Theta(\cdot,0))(z)$, $G_u(x) = \frac{1}{\sqrt{2\pi u}} e^{-\frac{x^2}{2u}}$

$$\Rightarrow \boxed{\Theta(z,u) = \frac{1}{\sqrt{2\pi u}} \int e^{-\frac{(z-y)^2}{2u}}\, \Theta(y,0)\, dy}$$

is the solution to the heat eq. for ini. cond. $\Theta(y,0)$.

$\Rightarrow$ with $\Theta(y,0) = \max(1-e^{-y},0)$, we get

$$\Theta(z,u) = \frac{1}{\sqrt{2\pi u}} \int_0^\infty e^{-\frac{(z-y)^2}{2u}} (1-e^{-y})\,dy$$

Now substituting back all changes of variables would indeed lead to the Black-Scholes formula that we discussed before: (we omit the details here)

$$C(S,0) = S\,\Phi(x) - Ke^{-rT}\,\Phi(x - \sigma\sqrt{T})$$

with cumulative normal distribution $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}}\,dy$

and $x = \dfrac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$