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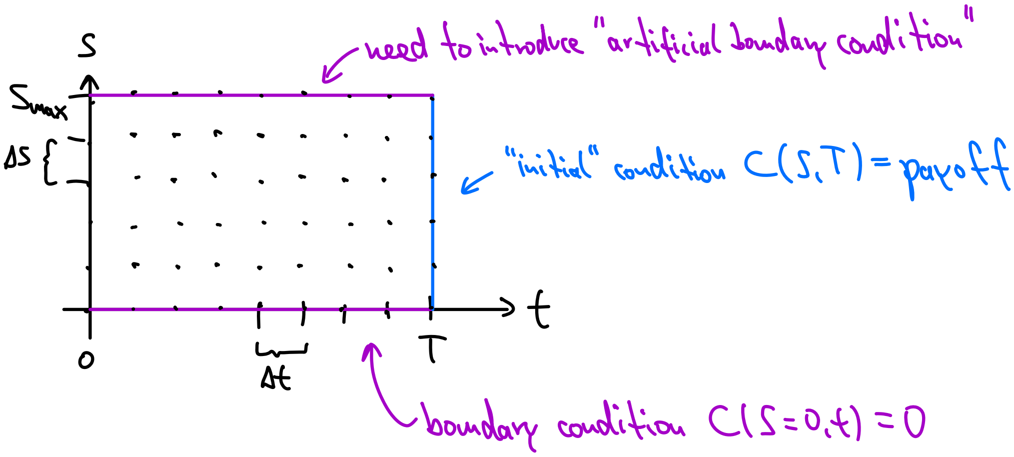
4.3 Discrete Finite Differences

In order to solve PDEs such as the Black-Scholes eq., we need to discretize S, t:



What do we use for $C(S_{max}, t)$? (for European calls)

↳ We know $C(S_{max}, T) = \max(S_{max} - K, 0) = S_{max} - K$ (S_{max} chosen large enough)

↳ Also: $C(S_{max}, 0) = S_{max} - Ke^{-rT}$, since risk vanishes as $S_{max} \to \infty$.

$\Rightarrow$ Could choose interpolation $\boxed{C(S_{max}, t) = S_{max} - Ke^{-r(T-t)}}$ as boundary condition.

Other (simpler) possibilities:

- $C(S_{max}, t) = S_{max} - K$
- $C(S_{max}, t) = S_{max}$, justified if $S_{max} \gg K$

We have discretized $[0, S_{max}] \times [0, T]$ into a grid:

- M steps of size $\Delta t = \frac{T}{M}$, $t_j = j \cdot \Delta t$
- N steps of size $\Delta S = \frac{S_{max}}{N}$, $S_i = i \cdot \Delta S$

We abbreviate $C(s_i, t_j) = C_i^j$ $\leftarrow$ time (j) $\leftarrow$ s (space) (i)

Then:

$$\boxed{\frac{\partial C_i^j}{\partial t} = \frac{C_i^{j+1} - C_i^j}{\Delta t} + \underbrace{O(\Delta t)}_{\text{terms of order } \Delta t \text{, i.e. } \frac{O(\Delta t)}{\Delta t} \xrightarrow{\Delta t \to 0} \text{const}}}$$

$$\left(\text{Taylor: } \underbrace{C(s_i, t_j + \Delta t)}_{C_i^{j+1}} = \underbrace{C(s_i, t_j)}_{C_i^j} + \frac{\partial C_i^j}{\partial t} \Delta t + \frac{1}{2} \frac{\partial^2 C_i^j}{\partial t^2} (\Delta t)^2 + O(\Delta t^3) \right)$$

For the s-derivative, we could choose

$$\frac{\partial C_i^j}{\partial s} = \frac{C_{i+1}^j - C_i^j}{\Delta s} + O(\Delta s)$$, but here one can do better

Taylor:

$$C(s_i + \Delta s) = C(s_i) + \frac{\partial C(s_i)}{\partial s} \Delta s + \frac{1}{2} \frac{\partial^2 C(s_i)}{\partial s^2} (\Delta s)^2 + \frac{1}{6} \frac{\partial^3 C_i}{\partial s^3} (\Delta s)^3 + O(\Delta s^4) \quad (1)$$

$$C(s_i - \Delta s) = C(s_i) - \frac{\partial C(s_i)}{\partial s} \Delta s + \frac{1}{2} \frac{\partial^2 C(s_i)}{\partial s^2} (\Delta s)^2 - \frac{1}{6} \frac{\partial^3 C_i}{\partial s^3} (\Delta s)^3 + O(\Delta s^4) \quad (2)$$

$$(1) - (2) \Rightarrow \quad C(s_i + \Delta s) - C(s_i - \Delta s) = 2 \frac{\partial C(s_i)}{\partial s} \Delta s + O(\Delta s^3)$$

$\Rightarrow$ The centered derivative $\boxed{\frac{\partial C_i^j}{\partial s} = \frac{C_{i+1}^j - C_{i-1}^j}{2 \Delta s} + O(\Delta s^2)}$ improves the error

Second derivative: $(1) + (2) \Rightarrow$ $\boxed{\frac{\partial^2 C_i^j}{\partial s^2} = \frac{C_{i+1}^j - 2C_i^j + C_{i-1}^j}{(\Delta s)^2} + O(\Delta s^2)}$

Same error as centralized first derivative