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6.4 Stability of Time-stepping Methods

Stability: Does the approximating/discretized solution converge to the true solution?

Let us just consider the simple example of exponential decay:

$$\frac{dy}{dt} = -\lambda y \quad , \lambda > 0 \quad \Rightarrow \quad y(t) = y_0 e^{-\lambda t}$$

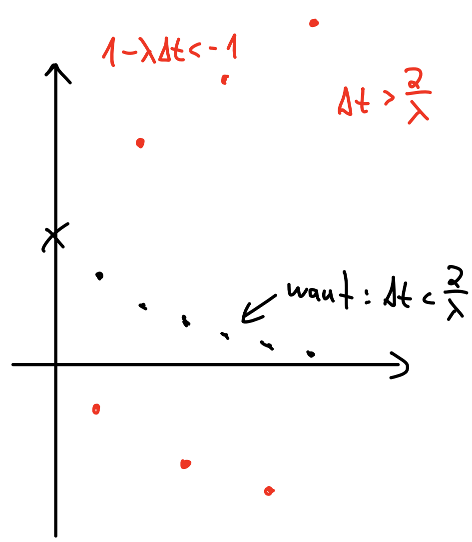
There are several ways to discretize the equation; we discuss the two most important ones:

- $\boxed{\text{Explicit Euler Method:}}$ $\dfrac{y^{j+1} - y^j}{\Delta t} = -\lambda y^j$ (rhs evaluated at j)

$$\Rightarrow \quad y^{j+1} = y^j - \lambda \Delta t \, y^j = (1 - \lambda \Delta t) y^j$$

$$\Rightarrow \quad y^M = (1 - \lambda \Delta t)^M y_0$$

$$\left(\text{check: } \lim_{M \to \infty} (1 - \lambda \underbrace{\tfrac{t}{M}}_{\Delta t})^M y_0 = e^{-\lambda t} y_0 \right)$$



In particular: Given y_0, we want the y^j to be decreasing:

So we need that $|1 - \lambda \Delta t| < 1$, i.e.,

- $1 - \lambda \Delta t < 1 \Rightarrow \lambda \Delta t > 0$ holds anyway
- $-(1 - \lambda \Delta t) < 1 \Rightarrow \lambda \Delta t < 2 \Rightarrow \underbrace{\Delta t < \frac{2}{\lambda}}_{\text{this is a condition on stability}}$

$\Rightarrow$ Here, we need to chose Δt small enough: s.t. $\Delta t < \frac{2}{\lambda}$

General conclusion: • easy to write down and to solve for y^{j+1} (in terms of y^j), even for eq.s such as $\frac{y^{j+1}-y^j}{\Delta t} = f(y^j)$

• there are usually stability conditions, e.g., Δt must be chosen small enough (or later, Δt must be small enough compared to Δs or Δx)

• **Implicit Euler Method:** $\frac{y^{j+1}-y^j}{\Delta t} = -\lambda y^{j+1}$ (rhs is evaluated at $j+1$)

$\Rightarrow y^{j+1} - y^j = -\lambda \Delta t \, y^{j+1} \Rightarrow y^{j+1} = \frac{1}{1+\lambda \Delta t} y^j$

$\Rightarrow y^n = \left(\frac{1}{1+\lambda\Delta t}\right)^n y_0$

So here, we need that $\left|\frac{1}{1+\lambda\Delta t}\right| < 1$, which always holds for $\lambda > 0$.

So here the scheme converges unconditionally!

If $\frac{y^{j+1}-y^j}{\Delta t} = f(y^{j+1})$ one needs to (locally) invert f

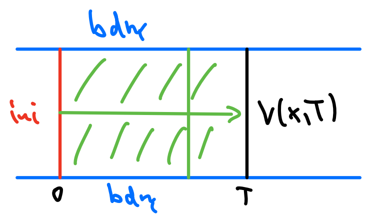
General conclusion: • might not be so easy to solve for y^{j+1} (in terms of y^j); and computationally less efficient than explicit schemes

• implicit schemes are usually unconditionally stable

4.5 Application to the Heat Equation

Heat eq.: $\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2}$, with initial value $V(x,0) = V_{ini}(x)$

say, with boundary conditions $V(x_0,t) = V_0$, $V(x_{max},t) = V_{max}$



(for B.-S. eq., it is the other way around: given $V(x,T)$, solve for $V(x,0)$)

$$\frac{\partial V}{\partial t}(x_i,t_j) = \frac{V(x_i,t_{j+1}) - V(x_i,t_j)}{\Delta t} + O(\Delta t) \quad \text{explicit}$$

$$\frac{\partial V}{\partial t}(x_i,t_{j+1}) = \frac{V(x_i,t_{j+1}) - V(x_i,t_j)}{\Delta t} + O(\Delta t) \quad \text{implicit}$$

eq.: $\frac{\partial V}{\partial t}(x,t) = \frac{\partial^2 V}{\partial x^2}(x,t)$

$$\frac{\partial^2 V}{\partial x^2}(x_i,t) = \frac{V(x_{i+1},t) - 2V(x_i,t) + V(x_{i-1},t)}{(\Delta x)^2} + O(\Delta x^2)$$

$$V(x_i,t_j) =: V_i^j$$

<u>Explicit scheme:</u> $\frac{V_i^{j+1} - V_i^j}{\Delta t} = \frac{V_{i+1}^j - 2V_i^j + V_{i-1}^j}{(\Delta x)^2}$ (here: given V^j, go to V^{j+1})

$$\Rightarrow V_i^{j+1} = \frac{\Delta t}{(\Delta x)^2} V_{i+1}^j + \left(1 - 2\frac{\Delta t}{(\Delta x)^2}\right) V_i^j + \frac{\Delta t}{(\Delta x)^2} V_{i-1}^j$$

we will see numerically that here $\frac{\Delta t}{(\Delta x)^2} < \text{const}$ is a condition for stability

<u>Implicit scheme:</u> $\frac{V_i^{j+1} - V_i^j}{\Delta t} = \frac{V_{i+1}^{j+1} - 2V_i^{j+1} + V_{i-1}^{j+1}}{(\Delta x)^2}$

$$V_i^j = -\underbrace{\frac{\Delta t}{(\Delta x)^2}}_{=:a} V_{i+1}^{j+1} + \left(1 + 2\frac{\Delta t}{(\Delta x)^2}\right) V_i^{j+1} - \frac{\Delta t}{(\Delta x)^2} V_{i-1}^{j+1}$$

Want to write this as $\vec{V}^j = A\,\vec{V}^{j+1}$

↳ vector in i

In matrix notation:

boundary condition still missing

$$\underbrace{\begin{pmatrix} (1+2a) & -a & & & 0 \\ -a & (1+2a) & -a & & \\ & -a & \ddots & \ddots & \\ & & \ddots & \ddots & -a \\ 0 & & & -a & (1+2a) \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} V_1^{j+1} \\ V_2^{j+1} \\ \vdots \\ V_n^{j+1} \end{pmatrix}}_{\vec{V}^{j+1}} = \underbrace{\begin{pmatrix} V_1^{j} \\ V_2^{j} \\ \vdots \\ V_n^{j} \end{pmatrix}}_{\vec{V}^{j}}$$

tridiagonal matrix

$\Longrightarrow$ write solution as $\vec{V}^{j+1} = A^{-1}\vec{V}^j$, i.e., need to invert A;

or equivalently, solve the system of linear equations $\vec{V}^j = A\vec{V}^{j+1}$, e.g., by Gaussian elimination

Boundary condition:

- $V(x_0, t) = V_0$, i.e., $V_0 = V_0^{j+1}$ $\forall j$:

$$V_1^j = -a\,V_2^{j+1} + (1+2a)\,V_1^{j+1} - a\,\underbrace{V_0^{j+1}}_{\text{given boundary condition}}$$

- $V(x_{max}, t) = V_{max}$, i.e., $V_{n+1}^{j+1} = V_{max}$ $\forall j$:

$$V_n^j = -a\,\underbrace{V_{n+1}^{j+1}}_{\text{given boundary condition}} + (1+2a)\,V_n^{j+1} - a\,V_{n-1}^{j+1}$$

So the right equation including the boundary conditions is:

$$\begin{pmatrix} (1+2a) & -a & & \\ -a & (1+2a) & \ddots & \\ & \ddots & \ddots & -a \\ & & -a & (1+2a)\end{pmatrix} \underbrace{\begin{pmatrix} V_1^{j+1} \\ V_2^{j+1} \\ \vdots \\ V_n^{j+1}\end{pmatrix}}_{\text{what I need to solve for}} = \underbrace{\begin{pmatrix} V_1^j + a V_0^{j+1} \\ V_2^j \\ \vdots \\ V_{n-1}^j \\ V_n^j + a V_{n+1}^{j+1}\end{pmatrix}}_{\text{given}}$$

given bdry conditions

or $$A \vec{V}^{j+1} = \vec{V}^j + \begin{pmatrix} a V_0^{j+1} \\ 0 \\ \vdots \\ 0 \\ a V_{n+1}^{j+1} \end{pmatrix}.$$