Jacobs University
Spring 2022

February 8, 2022

Advanced Calculus and Methods of Mathematical Physics

Homework 1

Due on February 15, 2022

Problem 1 [4 points]

Recall the mean value theorem of integral calculus: Suppose $g\colon [a,b] \to \mathbb{R}$ is Riemann integrable and non-negative, and $f\colon [a,b] \to \mathbb{R}$ is continuous. Then there exists $\xi \in [a,b]$ such that

$$f(\xi) \int_a^b g(x)\, \mathrm{d}x = \int_a^b f(x)\, g(x)\, \mathrm{d}x\,.$$

Give an example each to show that the following assumptions cannot be generally dropped:

(a) g is non-negative,

(b) f is continuous.

Problem 2 [6 points]

Recall Taylor's theorem in the following form. Suppose $f \in C^{n+1}(I)$ for some open interval I. Then for all $c, x \in I$,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}\,(x-c)^k + R_{n,c}(x)$$

where

$$R_{n,c}(x) = \frac{1}{n!} \int_c^x (x-t)^n\, f^{(n+1)}(t)\, \mathrm{d}t\,.$$

(a) Turn the derivation shown in class into a formal proof by induction.

(b) Show that the remainder can also be written as

$$R_{n,c}(x) = \frac{(x-c)^{n+1}}{n!} \int_0^1 (1-s)^n\, f^{(n+1)}(c + s(x-c))\, \mathrm{d}s\,.$$

(c) Show that there exists $\xi \in [c,x]$ such that

$$R_{n,c}(x) = \frac{(x-c)^{n+1}}{(n+1)!}\, f^{(n+1)}(\xi)\,.$$

(This expression is known as the *Lagrange form* of the remainder.)

Problem 3 [4 points]

Compute in a smart way the 4-th order Taylor polynomials around $c = 0$ of $f(x) = e^x \sin(2x)$ and $g(x) = e^{\sin x}$.

Problem 4 [6 points]

In class, we discussed the following version of the Fundamental Theorem of Calculus. Let $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$ and continuous at $\tilde{x} \in (a, b)$. Define $F : [a, b] \to \mathbb{R}$ by $F(x) - F(a) := \int_a^x f(t)\,\mathrm{d}t$. Then F is continuous on $[a, b]$ and differentiable at $\tilde{x}$ with $F'(\tilde{x}) = f(\tilde{x})$. Give a rigorous proof of this theorem.

Note: Recall the definitions of continuity and differentiability and try to start from there, using the assumptions of the theorem. If you are not so familiar with proofs, take a look at the Sloughter lecture notes (linked on the website), try to understand his proof, and write it down in your own words.