Jacobs University
Spring 2022

February 15, 2022

Advanced Calculus and Methods of Mathematical Physics

Homework 2

Due on February 22, 2022

Problem 1 [5 points]

Give a proof of the (convergence part of the) ratio test that was discussed in class, i.e., prove that $\sum_{k=0}^{\infty} a_k$ converges if $\lim_{k\to\infty} |a_{k+1}/a_k| < 1$. *Hint: Suppose $|a_{k+1}/a_k|$ converges to $r < 1$. Then, for large enough N, $|a_{N+1}/a_N| < R$ for some other $r < R < 1$. What can you conclude, using your knowledge about the geometric series?*

Problem 2 [3 points]

Use any of the convergence tests discussed in class and in the lecture notes to determine whether the following series converges:

(a) $\sum_{k=0}^{\infty} (-1)^k \frac{1}{k+1}$,

(b) $\sum_{k=1}^{\infty} \frac{1}{k^2}$,

(c) $\sum_{k=1}^{\infty} \frac{e^k}{k}$.

Problem 3 [2 points]

Determine the radius of convergence ρ of the following power series:

(a) $\sum_{k=0}^{\infty} x^k$,

(b) $\sum_{k=1}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$.

Problem 4 [5 points]

Compute the Taylor series of $f(x) = \ln(1+x)$ around $c = 0$, for $-1 < x < 1$. Show that the rest term indeed converges to 0 (for any $-1 < x < 1$). Does the Taylor series also converge for $x = 1$? Does is converge for $x = -1$? *Hint: Consider the integral form of the remainder.*

Problem 5 [5 points]

For any $\alpha \in \mathbb{R}$ on can define the following generalization of the binomial coefficient:

$$\binom{\alpha}{k} := \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}$$

for $k \geq 1$, and $\binom{\alpha}{0} := 1$. With that definition, let us define the function

$$f(x) = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

on the interval $(-1, 1)$.

(a) Show that

$$f'(x) = \frac{\alpha}{1+x} f(x) \,.$$

Justify the steps of your argument.

(b) Show that

$$g(x) = \frac{f(x)}{(1+x)^\alpha}$$

equals the constant 1.

(c) Conclude that $f(x) = (1+x)^\alpha$ on $(-1, 1)$.