Jacobs University — February 22, 2022
Spring 2022

Advanced Calculus and Methods of Mathematical Physics

Homework 3

Due on March 1, 2022

Problem 1 [4 points]

Let X and Y be metric spaces, $E \subset X$ compact, and $f : X \to Y$ continuous. Show that $f(E) := \{f(x) \in Y : x \in E\}$ is a compact subset of Y.

Problem 2 [4 points]

Use the definition of compactness from class (i.e., E is compact if every open cover of E has a finite subcover) to show that $E = (0, 1) \subset \mathbb{R}$ is not compact. (We know this from Heine–Borel, but here the exercise is to show it starting from the definition.)

Problem 3 [4 points]

The operator norm for linear maps A between normed vector spaces X and Y is defined by

$$\|A\| := \sup_{x \in X, \|x\|=1} \|Ax\| .$$

(a) Verify that the operator norm is indeed a norm.

(b) Let X, Y, and Z be normed vector spaces, and let $B \colon X \to Y$ and $A \colon Y \to Z$ be linear maps. Show that

$$\|AB\| \leq \|A\| \, \|B\| .$$

Problem 4 [4 points]

Draw the following unit balls in a two-dimensional coordinate system:

(a) $B_1 := \{y \in \mathbb{R}^2 : \|y\|_1 \leq 1\}$,

(b) $B_2 := \{y \in \mathbb{R}^2 : \|y\|_2 \leq 1\}$,

(c) $B_3 := \{y \in \mathbb{R}^2 : \|y\|_4 \leq 1\}$,

(d) $B_\infty := \{y \in \mathbb{R}^2 : \|y\|_\infty \leq 1\}$.

Recall that

$$\|y\|_p := \left(\sum_i |x_i|^p \right)^{1/p}.$$

Problem 5 [4 points]

We denote by X the set of all real-valued continuous functions on the interval $[a, b]$. For $f, g \in X$, we define

$$d_1(f, g) := \int_a^b |f(x) - g(x)| \, \mathrm{d}x, \quad d_\infty(f, g) := \max_{x \in [a,b]} |f(x) - g(x)|.$$

(a) Show that (X, d_1) and (X, d_∞) are metric spaces.

(b) Show that if $f_n \to f$ in (X, d_∞), then also $f_n \to f$ in (X, d_1).

(c) Show that there is a sequence (f_n) in X that converges to some f in (X, d_1) but that does not converge to f in (X, d_∞).