Jacobs University
Spring 2022

March 22, 2022

Advanced Calculus and Methods of Mathematical Physics

Homework 7

Due on March 29, 2022

Problem 1 [4 points]

(Kantorovitz, p. 175, Exercise 1.) Let $F(y) = \int_0^1 e^{x^2 y} \, \mathrm{d}x$.

(a) Find $F'(0)$.

(b) For $y \neq 0$, show that F satisfies the differential equation

$$2y \, F'(y) + F(y) = e^y \, .$$

Problem 2 [5 points]

(Kantorovitz, p. 175, Exercise 1.) Let $b > 0$. For $f \in C([0,b])$, define $F_0(x) = f(x)$ and

$$F_n(x) = \frac{1}{(n-1)!} \int_0^x (x-y)^{n-1} \, f(y) \, \mathrm{d}y$$

for $x \in [0,b]$ and $n = 1, 2, \ldots$. Show that $F_n \in C^n([0,b])$ with

$$F_n^{(k)} = F_{n-k} \quad \text{for } k = 1, \ldots, n \, .$$

Remark: This relation shows that if J denotes the *integration operator* on $C([0,b])$ defined by

$$(Jf)(x) = \int_0^x f(y) \, \mathrm{d}y \, ,$$

then

$$F_n = J^n f \, .$$

Problem 3 [5 points]

(Kantorovitz, p. 177, Exercise 5) Let $0 < a < b$ and

$$F(y) = \int_{a+y}^{b+y} \frac{e^{xy}}{x} \, \mathrm{d}x \, .$$

Calculate $F'(y)$ for $y > 0$.

Problem 4 [6 points]

(Adapted from Kantorovitz, Example 4.1.12) Consider the function

$$f(x,y) = \frac{x^2 - y^2}{(x^2+y^2)^2}$$

on the rectangle $I = [0,1] \times [\alpha, 1]$ for $\alpha \in (0,1)$.

(a) Show that

$$\int_0^1 f(x,y)\ \mathrm{d}x = -\frac{1}{1+y^2}$$

for every fixed $y \in [\alpha, 1]$.

Hint: Note that

$$\frac{x^2 - y^2}{(x^2+y^2)^2} = \frac{1}{x^2+y^2} + y\,\frac{\partial}{\partial y}\frac{1}{x^2+y^2}\,.$$

(b) Note that the result from (a) continuously extends to the unit square $I = [0,1]^2$ and conclude that

$$\int_0^1 \int_0^1 f(x,y)\ \mathrm{d}x\ \mathrm{d}y = -\frac{\pi}{4}$$

while

$$\int_0^1 \int_0^1 f(x,y)\ \mathrm{d}y\ \mathrm{d}x = \frac{\pi}{4}\,.$$

Bonus Problem [4 points]

Prove the Heine–Cantor theorem: If (K, d_1) and (Y, d_2) are metric spaces, K is compact, and $f : K \to Y$ is continuous, then f is uniformly continuous. (*Note:* In the lecture notes, a proof was given using sequential compactness. Here, the task is to prove the theorem directly from the definition of compactness ("Every open cover has a finite subcover.").)