Jacobs University
Spring 2022

March 29, 2022

Advanced Calculus and Methods of Mathematical Physics

Homework 8

Due on April 5, 2022

Problem 1 [3 points]

(Kantorovitz, p. 177, Exercise 7) For $n \in \mathbb{N}$, calculate the iterated integral

$$\int_0^\pi \int_0^1 x^{2n-1} \cos(x^n y) \, \mathrm{d}x \, \mathrm{d}y \, .$$

Problem 2 [3 points]

Let $B \subset \mathbb{R}^n$ be a bounded set and define the *characteristic function* of B by

$$\chi_B(x) = \begin{cases} 1 & \text{if } x \in B \, , \\ 0 & \text{if } x \notin B \, . \end{cases}$$

Show that χ_B is Riemann-integrable if and only if B has content.

Problem 3 [6 points]

(Kantorovitz, p. 177, Exercise 8)

(a) Calculate the iterated integral

$$\int_0^1 \int_0^1 \frac{x}{(1+x^2)(1+xy)} \, \mathrm{d}x \, \mathrm{d}y$$

in two different ways, and prove thereby that

$$\int_0^1 \frac{\ln(1+x)}{1+x^2} \, \mathrm{d}x = \frac{\pi \ln 2}{8} \, .$$

(b) Conclude that

$$\int_0^1 \frac{\arctan x}{1+x} \, \mathrm{d}x = \frac{\pi \ln 2}{8} \, .$$

Problem 4 [4 points]
Let $D \subset \mathbb{R}^2$ be the domain bounded by the parabola $x = y^2$ and the line $x = y$. Compute

$$\int_D \sin \frac{\pi x}{y} \, \mathrm{d}S \, .$$

Problem 5 [4 points]
Let $D \subset \mathbb{R}^2$ be the annulus with radii $0 < a < b$, i.e., $D := \{(x, y) \in \mathbb{R}^2 : a \leq \sqrt{x^2 + y^2} \leq b\}$. Compute

$$\int_D \arctan \frac{y}{x} \, \mathrm{d}S \, .$$