Jacobs University
Spring 2022

April 26, 2022

Advanced Calculus and Methods of Mathematical Physics

Homework 11

Due on May 3, 2022

Problem 1 [6 points]

(Kantorovitz, Exercises 4.4.5, Problems 1, 2, 3.) Find the area $S(D)$ of the domain $D \subset \mathbb{R}^2$ with given boundary $\partial D = \gamma$ for the cases where:

(a) γ is the *cardioid* parameterized by

$$\gamma(t) = (1 - \cos t)(\cos t, \sin t)\,, \quad t \in [0, 2\pi].$$

(b) γ is the *hypocycloid* satisfying the equation

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$$

with $a, b > 0$. (*Hint:* Find a trigonometric parametrization.)

(c) γ is the curve with parametrization

$$\gamma(t) = (t^2 - 1)(1, t)\,, \quad t \in [-1, 1]\,.$$

Problem 2 [7 points]

(From Moskowitz/Paliogiannis, p. 460.) Evaluate the following line integrals along a curve γ, both directly and by using Green's theorem:

(a) $\displaystyle\int_\gamma \Big[(1-x^2)\,y\;\mathrm{d}x + (1+y^2)\,x\;\mathrm{d}y\Big]$, where γ is the unit circle in anti-clockwise orientation.

(b) $\displaystyle\int_\gamma \Big[x\,y^2\;\mathrm{d}y - x^2\,y\;\mathrm{d}x\Big]$, where γ is the boundary of the annulus $1 \le x^2 + y^2 \le 4$, in standard orientation (i.e., counter-clockwise for the outer circle and clockwise for the inner circle).

(c) $\displaystyle\int_\gamma \Big[(y - \sin x)\;\mathrm{d}x + \cos x\;\mathrm{d}y\Big]$, where γ is the perimeter of the triangle with vertices $(0, 0)$, $(\frac{\pi}{2}, 0)$, and $(\frac{\pi}{2}, 1)$ in counter-clockwise orientation.

Problem 3 [7 points]
Prove the following identities for vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \in \mathbb{R}^3$.

(a) The *"BAC–CAB-identity"*

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}\,(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}\,(\boldsymbol{a} \cdot \boldsymbol{b})\,.$$

(b) The *Jacobi identity* in three dimensions

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) + \boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{a}) + \boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) = 0\,.$$

(c) The *Cauchy–Binet formula* in three dimensions

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})\,(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d})\,(\boldsymbol{b} \cdot \boldsymbol{c})\,.$$

Then, show that $\|\boldsymbol{a} \times \boldsymbol{b}\| = \|\boldsymbol{a}\|\|\boldsymbol{b}\| \sin\theta$ (which equals the area of a parallelogram spanned by $\boldsymbol{a}$ and $\boldsymbol{b}$), where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.