Jacobs University May 3, 2022
Spring 2022

Advanced Calculus and Methods of Mathematical Physics

Homework 12

Due on May 10, 2022

Problem 1 [5 points]

(Kantorovitz, Exercises 4.5.7, Problem 6.) A smooth curve is given in the yz-plane by the parameterization

$$\gamma(t) = (0, y(t), z(t)) \,, \quad t \in [a, b] \,.$$

The surface M is obtained by revolving γ about the z-axis.

(a) Show that M has surface area

$$\sigma(M) = 2\pi \int_\gamma |y| \, \mathrm{d}s = 2\pi \int_a^b |y(t)| \, \|\gamma'(t)\| \, \mathrm{d}t \,.$$

(b) Take γ to be the circle centered at $(0, R, 0)$ with radius $r \in (0, R)$ in the yz-plane, so that M is a torus. Find the area of the torus M.

Problem 2 [5 points]

Show, by explicit computation, that the surface area of a smooth regular surface M with parameterization $f \in C^1(U \to \mathbb{R}^3)$,

$$\sigma(M) = \int_U \|n\| \, \mathrm{d}S = \int_U \left\| \frac{\partial f}{\partial u_1} \times \frac{\partial f}{\partial u_2} \right\| \mathrm{d}u$$

is independent of the parameterization. I.e., if $g \in C^1(V, \mathbb{R}^3)$ is another smooth regular parameterization with $g = f \circ \phi$ for some $\phi \in C^1(V, U)$, then

$$\sigma(M) = \int_V \left\| \frac{\partial g}{\partial v_1} \times \frac{\partial g}{\partial v_2} \right\| \mathrm{d}v \,.$$

Hint: Use chain rule, change-of-variable formula, and the properties of the cross-product.

Problem 3 [5 points]

(Kantorovitz, Exercises 4.5.7, Problem 4.) Let $F = (xy, 0, -z^2)$, $D = [0, 1]^3$, and $M = \partial D$ oriented such that the normal vector points outwards. Calculate the flux

$$\Phi = \int_M F \cdot n \, \mathrm{d}\sigma$$

(a) by applying the divergence theorem,

(b) directly.

Problem 4 [5 points]

(Kantorovitz, Exercises 4.5.7, Problem 7.) Let γ be the closed curve parameterized by

$$\gamma(t) = (\cos t, \sin t, \cos 2t)\,, \quad t \in [0, 2\pi]\,.$$

Let M be the portion of the hyperbolic paraboloid S defined by the equation

$$z = x^2 - y^2$$

with boundary γ (note that γ lies on S!). Calculate the line integral

$$\int_\gamma F \cdot \mathrm{d}x$$

for the vector field $F = (x^2 + z^2, y, z)$

(a) by applying Stokes' theorem,

(b) directly.