Advanced Calculus and Methods of Mathematical Physics — Session 2

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For the Riemann integral, let us consider $f: [a,b] \to \mathbb{R}$ (bounded). We define:

- a partition $\mathcal{P}$ of $[a,b]$ is a finite set of points $x_1, \ldots, x_n$ s.t. $a = x_0 \leq x_1 \leq \ldots \leq x_{n-1} \leq x_n = b$;
- the upper Riemann sum is $U(f,\mathcal{P}) := \sum_{i=1}^{n} \left[\sup_{x \in [x_{i-1}, x_i]} f(x) \right] (x_i - x_{i-1})$;
- the lower Riemann sum is $L(f,\mathcal{P}) := \sum_{i=1}^{n} \left[\inf_{x \in [x_{i-1}, x_i]} f(x) \right] (x_i - x_{i-1})$;
- the upper Riemann integral $\overline{\int_a^b} f(x)dx := \inf_{\mathcal{P}} U(f,\mathcal{P})$ ← always defined
- the lower Riemann integral $\underline{\int_a^b} f(x)dx := \sup_{\mathcal{P}} L(f,\mathcal{P})$ ← always defined

[draw pictures to illustrate these definitions]

[do you recall a counter example? i.e., a function that is not Riemann integrable]

Then: $f: [a,b] \to \mathbb{R}$ is **Riemann integrable** if upper and lower Riemann integrals coincide.

$$\text{Then } \int_a^b f \equiv \int_a^b f(x)dx := \underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx.$$

$$\equiv \int_a^b dx\, f(x)$$

Recall a few standard results:

- f differentiable at $\tilde{x} \Rightarrow f$ continuous at $\tilde{x}$

(but converse does not hold; there are even everywhere continuous and nowhere differentiable fct.s)

- product (or Leibniz) rule and chain rule
- mean-value thm.: Let $f: [a,b] \to \mathbb{R}$ be cont. and differentiable on (a,b). Then there is a $z \in (a,b)$ with

$$f'(z) = \frac{f(b) - f(a)}{b-a}$$

[draw a picture to visualize this]

- $f:[a,b]\to\mathbb{R}$ continuous $\Rightarrow$ f integrable
- Fundamental Theorem of Calculus:
  - version 1: Let f be integrable on $[a,b]$, F cont. on $[a,b]$ and differentiable on (a,b) with $F'(x)=f(x)$ $\forall x\in(a,b)$ (such F are called anti-derivative of f). Then
  $$\int_a^b f(x)\,dx = F(b)-F(a).$$
  - version 2: Let f be integrable on $[a,b]$ and continuous at $\tilde{x}\in(a,b)$. Define $F:[a,b]\to\mathbb{R}$ by $F(x)-F(a):=\int_a^x f(t)\,dt$. Then F is continuous on $[a,b]$ and differentiable at $\tilde{x}$ with $F'(\tilde{x})=f(\tilde{x})$. [can you prove this?]
- integration by parts, integration by substitution

Let us recall the following notation. Let $f:I\to\mathbb{R}$, I an open interval. Then:

f is of class $C^n(I)$ if f is n times differentiable and all derivatives $f', f'', \dots, f^{(n)}$ are cont.

Next: Taylor series. Let $f:I\to\mathbb{R}$, I an open interval.

We try to approximate $f(x)$ near $c\in I$ by polynomials s.t. the derivatives up to some order agree in c.

For f n times differentiable, we write $f(x)=P_{n,c}(x)+R_{n,c}(x)$ with
$P_{n,c}(x):=\sum_{k=0}^{n}\frac{f^{(k)}(c)}{k!}(x-c)^k$ the n-th order Taylor polynomial about c, and

$R_{n,c}(x)$ the n-th order Taylor remainder.

[check that indeed $P_{n,c}^{(\ell)}(c)=f^{(\ell)}(c)$ $\forall\, \ell=0,\dots,n$]

Ex.: For $f(x)=e^x$, we have $P_{n,0}(x)=\sum_{k=0}^{n}\frac{x^k}{k!}$.

Note: We know that $\lim_{x\to c}\frac{R_{n,c}(x)}{(x-c)^n}=0$ (e.g., by using L'Hôpital n times).

[check this]

Notation: $R_{n,c}(x)=o((x-c)^n)$ as $x\to c$. If $\boxed{h(x)=o(g(x)) \text{ as } x\to c}$ ($c=\pm\infty$ possibly), we say $h(x)$ is "of smaller order than $g(x)$" as $x\to c$ (or "little oh of g of x") if $\frac{h(x)}{g(x)}\to 0$ as $x\to c$.

Note: There is also the "big oh" notation: $\boxed{h(x)=O(g(x))}$ if $\left|\frac{h(x)}{g(x)}\right|\le$ constant as $x\to c$.

"h is of the order of g"