Advanced Calculus and Methods of Mathematical Physics — Session 4

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Recall: $(f_n)_n$ with $f_n: D \to \mathbb{R}$, $D \subset \mathbb{R}$ converges uniformly to $f: D \to \mathbb{R}$ if

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N \text{ and } \forall x \in D: \ |f_n(x) - f(x)| < \varepsilon.$$

Then indeed we have the following result:

Theorem:

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions with $f_n: D \to \mathbb{R}$, $D \subset \mathbb{R}$, and $f: D \to \mathbb{R}$. Then:

a) If $f_n \to f$ uniformly and all f_n are continuous, then f is continuous.

b) If $f_n \to f$ uniformly, $D = [a,b]$ for some $a<b$ and all f_n are Riemann integrable, then f is Riemann integrable and $\lim\limits_{n\to\infty} \int_a^b f_n = \int_a^b f$.

c) If $D = [a,b]$ for some $a<b$ and all f_n are $\underbrace{C^{(1)}}_{\text{continuously differentiable}}$, $g: D \to \mathbb{R}$, and $f_n \to f$ pointwise, $f_n' \to g$ uniformly, then g is continuous and f is $C^{(1)}$ with $f' = g$.

Proof of b): If $f_n \to f$ uniformly, then $\varepsilon_n := \sup\limits_{x\in[a,b]} |f_n(x) - f(x)| \xrightarrow{n\to\infty} 0$.

So for any $n \in \mathbb{N}$: $\quad f_n - \varepsilon_n \leq f \leq f_n + \varepsilon_n \quad$ (these inequalities are meant pointwise)

$$\Rightarrow \int_a^b (f_n - \varepsilon_n) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq \int_a^b (f_n + \varepsilon_n), \quad (*)$$

so $\quad 0 \leq \overline{\int_a^b} f - \underline{\int_a^b} f \leq 2(b-a)\varepsilon_n \xrightarrow{n\to\infty} 0$, so f Riemann integrable.

$$\Rightarrow \overline{\int_a^b} f = \underline{\int_a^b} f = \int_a^b f \text{ and } \underset{\text{in } (*)}{\left| \int_a^b f - \int_a^b f_n \right|} \leq (b-a)\varepsilon_n \xrightarrow{n\to\infty} 0, \text{ so } \int_a^b f_n \to \int_a^b f. \quad \square$$

Note:

- The theorem in particular applies to $\underbrace{S_n := \sum_{k=0}^{n} f_k}_{\text{called "partial sums"}}$, e.g., if $\sum_{k=0}^{n} f_k \xrightarrow{n\to\infty} S \equiv \sum_{k=0}^{\infty} f_k$ uniformly and all f_k Riemann integrable, then $\sum_{k=0}^{\infty} \int_a^b f_k = \int_a^b \sum_{k=0}^{\infty} f_k$.
- One can show: If $|f_k(x)| \leq M_k$ $\forall k \, \forall x$ with $\sum_{k=0}^{\infty} M_k$ convergent, then $\sum_{k=0}^{n} f_k$ converges uniformly to f, def. pointwise by $f(x) = \sum_{k=0}^{\infty} f_k(x)$. (Weierstrass M-test)

1.3 Power Series

First, a quick review on convergence of series.

Let $(a_k)_k$ be a sequence in $\mathbb{R}$ and $s_n := \sum_{k=0}^{n} a_k$ the partial sums. Then:

- $\sum_{k=0}^{\infty} a_k$ exists $:\Longleftrightarrow \lim_{n\to\infty} \sum_{k=0}^{n} a_k$ exists
- $\sum_{k=0}^{\infty} a_k$ converges absolutely $:\Longleftrightarrow \sum_{k=0}^{\infty} |a_k|$ converges

The most common convergence tests are:

- Comparison test: Fix some $N \in \mathbb{N}$. Then:
  - If $|a_k| \leq b_k$ $\forall k \geq N$ and $\sum b_k$ conv., then $\sum a_k$ conv.
  - If $a_k \geq c_k \geq 0$ $\forall k \geq N$ and $\sum c_k$ diverges, then $\sum a_k$ diverges.
- Root test: Define $L := \limsup_{k\to\infty} \sqrt[k]{|a_k|}$. Then:
  - If $L < 1$: $\sum a_k$ conv.,
  - If $L > 1$: $\sum a_k$ diverges,
  - If $L = 1$ the test is inconclusive.

Can be applied to any sequence!
(no specific conditions like in the other tests)

- Ratio test: If $\limsup_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| < 1$, then $\sum a_k$ conv.

- Integral test: Let $f:[0,\infty)\to[0,\infty)$ be Riemann integrable on $[0,b]$ $\forall b>0$ and monotonically decreasing. Then $\sum_{k=0}^{\infty} f(k)$ exists if and only if $\int_0^{\infty} f(x)dx$ exists.

- Leibniz criterion ("alternating series test"): Let $a_k \geq 0$ and (a_k) monotonically decreasing with $a_k \xrightarrow{k\to\infty} 0$. Then $\sum (-1)^k a_k$ converges.

Examples: see homework