Advanced Calculus and Methods of Mathematical Physics — Session 5

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We consider a sequence $\sum_{k=0}^{\infty} a_k$ in $\mathbb{R}$. An important convergence test is:

- Root test: Define $L := \limsup_{k\to\infty} \sqrt[k]{|a_k|}$. Then:
  - If $L<1$: $\sum a_k$ conv.,
  - If $L>1$: $\sum a_k$ diverges,
  - If $L=1$ the test is inconclusive.

Note: For a sequence (a_n) in $\mathbb{R}$ we def.:

- $\limsup_{k\to\infty} a_k := \inf_{n\in\mathbb{N}} \left(\sup_{k\geq n} a_k \right)$
- $\liminf_{k\to\infty} a_k := \sup_{n\in\mathbb{N}} \left(\inf_{k\geq n} a_k \right)$

E.g., $\lim_{k\to\infty} (-1)^k$ does not exist, but $\limsup_{k\to\infty} (-1)^k = 1$ and $\liminf_{k\to\infty} (-1)^k = -1$.

Next: We define the Cauchy product of $\sum a_k$ and $\sum b_k$ by $\sum c_k$ with $c_k = \sum_{\ell=0}^{k} a_\ell b_{k-\ell}$.

$\left((a_0+a_1+a_2+\dots)(b_0+b_1+b_2+\dots) = a_0b_0 + (a_1b_0+a_0b_1) + (a_2b_0+a_1b_1+a_0b_2) + \dots \right)$

Result: If $\sum a_k$ conv. to A and $\sum b_k$ conv. to B, and at least one of the series conv. absolutely, then $\sum_{k=0}^{\infty}\sum_{\ell=0}^{k} a_\ell b_{k-\ell}$ conv. absolutely to AB.

Then, finally, let us consider a (complex) $\boxed{\text{power series}}$ $\underbrace{\sum_{k=0}^{\infty} c_k z^k}$, with $c_k, z \in \mathbb{C}$.

more general, we could consider $\sum_{k=0}^{\infty} c_k (z-c)^k$

From the root test we get that it converges if

$$\limsup_{k\to\infty} \sqrt[k]{|c_k||z|^k} = \limsup_{k\to\infty} \sqrt[k]{|c_k|}\, |z| < 1 \quad \text{i.e., if } |z| < \boxed{\frac{1}{\limsup_{k\to\infty} \sqrt[k]{|c_k|}} =: \rho}.$$

This ρ is called $\boxed{\text{radius of convergence}}$ (possibly 0 or ∞).

Alternatively: Whenever ratio test applies, then $\rho = \lim_{k\to\infty} \left|\frac{c_k}{c_{k+1}}\right|$.

So for all $|z| < \rho$ (i.e., on an open disc with radius ρ), $\sum c_k z^k$ conv. absolutely, and for $|z| > \rho$ it diverges. (For $|z| = \rho$ it might or might not converge.)

In fact: $\sum c_k z^k$ converges uniformly on the set $\{z \in \mathbb{C} : |z| \leq \rho - \varepsilon\}$ for any $\varepsilon > 0$.

For a real power series $f(x) = \sum a_k x^k$, the previous thm. says:

- f is differentiable on $(-\rho, \rho)$ and $f'(x) = \sum k a_k x^{k-1}$,
- for any $[0,x] \subset (-\rho,\rho)$ we have $\int_0^x f(t)\,dt = \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1} + \text{const.}$

we can take derivatives and integrate term-wise

Ex.: $\sum_k \frac{x^k}{k!}$ has radius of convergence $\rho = \lim_{k\to\infty} \frac{1/k!}{1/(k+1)!} = \lim_{k\to\infty} (k+1) = \infty$, so it converges $\forall x \in \mathbb{R}$, and $\frac{d}{dx} \sum_{k=0}^{\infty} \frac{x^k}{k!} = \sum_{k=0}^{\infty} k \frac{x^{k-1}}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ (i.e., $\frac{d}{dx} e^x = e^x$).

1.4 Metric and Normed Spaces

Let us take an abstract point of view for a moment.

Given some set X, there are different possibilities to define "closeness" for elements of X:

- The most general/abstract type of space is a topological space (X, τ), where τ is a collection of subsets of X satisfying the axioms:
  - $\emptyset \in \tau$, $X \in \tau$,
  - arbitrary unions of elements of τ belong to τ,
  - finite intersections of elements of τ belong to τ.

  Elements of τ are called "open sets".

  This allows us to define, e.g.:
  - convergence (for each $\underbrace{\text{neighborhood } U \text{ of } x}_{\text{any open set that contains } x}$, $\exists N \in \mathbb{N}$ s.t. $x_n \in U$ $\forall n \geq N$)
  - continuity (preimages of open sets are open)
  - compactness (every open cover has a finite subcover)
  - connectedness

  However, general topological spaces do not necessarily quantify "closeness".

  So next time, we will introduce metric, normed, and inner product spaces.