Advanced Calculus and Methods of Mathematical Physics — Session 6

Prof. Sören Petrat, Jacobs University, Spring 2022 — Feb. 18, 2022

We continue our discussion of structures that define "closeness" of two elements in a set:

- **Metric spaces** (M,d) have a distance function $d: M\times M \to [0,\infty)$ called a metric.

  A fct. $d: M\times M \to [0,\infty)$ is called a **metric** if:

  - $d(x,y) = 0 \iff x = y$ (definiteness),
  - $d(x,y) = d(y,x)$ $\forall x,y\in M$ (symmetry),
  - $d(x,y) \leq d(x,z) + d(z,y)$ $\forall x,y,z \in M$ (triangle inequality).

  Let us define the open ball around x with radius r as $B_r(x) := \{y\in M : d(x,y) < r\}$.

  Then arbitrary unions of open balls can be def. as the open sets of a topological space (they induce the "metric topology").

  (Every metric space defines a topological space, but not every topological space is induced by a metric.)

- More concretely: Any vector space V over $\mathbb{R}$ (or $\mathbb{C}$ or any other field) together with a **norm** $\|\cdot\|: V\to[0,\infty)$, $x\mapsto \|x\|$ is called **normed space**. A norm is def. by:

  - $\|x\| = 0 \iff x = 0$ (definiteness),
  - $\|\lambda x\| = |\lambda|\,\|x\|$ $\forall \lambda\in\mathbb{R}$ (or $\mathbb{C}$), $x\in V$ (homogeneity),
  - $\|x+y\| \leq \|x\| + \|y\|$ $\forall x,y\in V$ (triangle inequality).

  Any norm defines a metric via $d(x,y) = \|x-y\|$. (But not every metric comes from a norm.)

- Even more special are $\boxed{\text{inner product spaces}}$. A map $\langle\cdot,\cdot\rangle : V\times V \to \mathbb{R}$, $(x,y)\mapsto \langle x,y\rangle = x\cdot y$ is called $\boxed{\text{inner product}}$ if:
  - $\langle x, \alpha y+\beta z\rangle = \alpha\langle x,y\rangle + \beta\langle x,z\rangle \quad \forall x,y,z\in V, \alpha,\beta\in\mathbb{C}$ (linearity),
  - $\langle x,y\rangle = \overline{\langle y,x\rangle} \quad \forall x,y\in V$ (conjugate symmetry),
  - $\langle x,x\rangle \geq 0$, and $\langle x,x\rangle = 0 \Leftrightarrow x=0$ (positive definitness).

  Any inner product defines a norm via $\|x\| = \sqrt{\langle x,x\rangle}$. (But not every norm comes from an inner product.)

  Note: An important inequality is $\boxed{\text{Cauchy-Schwarz}}$: $|\langle x,y\rangle| \leq \|x\|\,\|y\| \quad \forall x,y\in V$.

Conceptually:
- topologies define "closeness",
- metrics define "distance",
- norms define "length",
- inner products define "angles".

Let us exemplify the above for $\mathbb{R}^n$. We denote $x=(x_1,\dots,x_n)\in\mathbb{R}^n$. Then:

- The inner product (or scalar product) is $\langle x,y\rangle \equiv x\cdot y := \sum_{i=1}^n x_i y_i$.

  (or dot product)

  (Recall that $\langle x,y\rangle = \|x\|\,\|y\| \underbrace{\cos(\sphericalangle x,y)}_{\text{angle between } x \text{ and } y}$.)

- It induces the norm $\|x\|_2 = (\langle x,x\rangle)^{\frac{1}{2}} = \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}$.
- Also $\|x\|_p := \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}$ define norms $\forall\, 1\leq p<\infty$ and the following Hölder inequality holds: $|\langle x,y\rangle| \leq \|x\|_p\,\|y\|_q$.
- Also $\|x\|_\infty := \max_{1\leq i\leq k} |x_i|$ defines a norm. (In fact, $\lim_{p\to\infty}\|x\|_p = \|x\|_\infty$.)
- In fact, all norms on $\mathbb{R}^n$ are equivalent, i.e., $\forall x\ \exists C_1,C_2>0$ s.t. $C_1\|x\|_A \leq \|x\|_B \leq C_2\|x\|_A$

  $\|\cdot\|_A$ and $\|\cdot\|_B$ are any two norms here

Finally, we briefly discuss the notion of compactness.

Recall that we call a subset of $\mathbb{R}^n$ open if it can be written as the union of open balls

$$B_r(x) := \{ y \in \mathbb{R}^n : \underbrace{\|x-y\|}_{\text{unless specified otherwise, we always mean } \|x\|^2 := \sum_{i=1}^n x_i^2 \text{ for } x \in \mathbb{R}^n} \leq r \}.$$