Advanced Calculus and Methods of Mathematical Physics Session 11

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Last time, we introduced the Hessian H_f of a function $f: U \to \mathbb{R}$ ($U \subset \mathbb{R}^n$ open) as $\boxed{(H_f)_{ij} := \frac{\partial^2 f}{\partial x_i \partial x_j}}$. For $f \in C^2$, we found $(H_f)_{ij} = (H_f)_{ji}$ (Clairaut, Schwarz).

The Hessian can be used to determine whether an extremum is a maximum or minimum:

Theorem: Let $f: U \to \mathbb{R}$, $U \subset \mathbb{R}^n$ open, $f \in C^2(U)$, with $(\nabla f)(x) = 0$ for some $x \in U$.
Then:

- If $H_f(x)$ is positive definite (i.e., $\langle h, H_f(x) h \rangle > 0 \;\forall h \in \mathbb{R}^n, h \neq 0$), then f has a local minimum at x.
- If $H_f(x)$ is negative definite (i.e., $\langle h, H_f(x) h \rangle < 0 \;\forall h \in \mathbb{R}^n, h \neq 0$), then f has a local maximum at x.

Proof: Follows from the Taylor expansion (making h very small). $\square$

Note: Since H is symmetric, all eigenvalues are real. Then H is positive definite if and only if all eigenvalues are positive.

A very simple example: $f(x,y) = -x^2 - y^2$.

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} -2x \\ -2y \end{pmatrix}. \quad \nabla f = 0 \text{ for } (x,y) = (0,0).$$

$$H_f((x,y) = (0,0)) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}, \text{ so } H_f(0) \text{ is negative definite.}$$

$\Rightarrow f$ has a maximum at $(0,0)$.

Other examples (see geogebra pictures below):

- $f(x,y) = x^2 - y^2 + 2 \implies H_f(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \implies$ saddle point

  ($\langle h, H_f h \rangle > 0$ for some h, and $\langle h, H_f h \rangle < 0$ for others)

- $f(x,y) = x^3 - y^2 + 2 \implies H_f(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \implies$ degenerate point

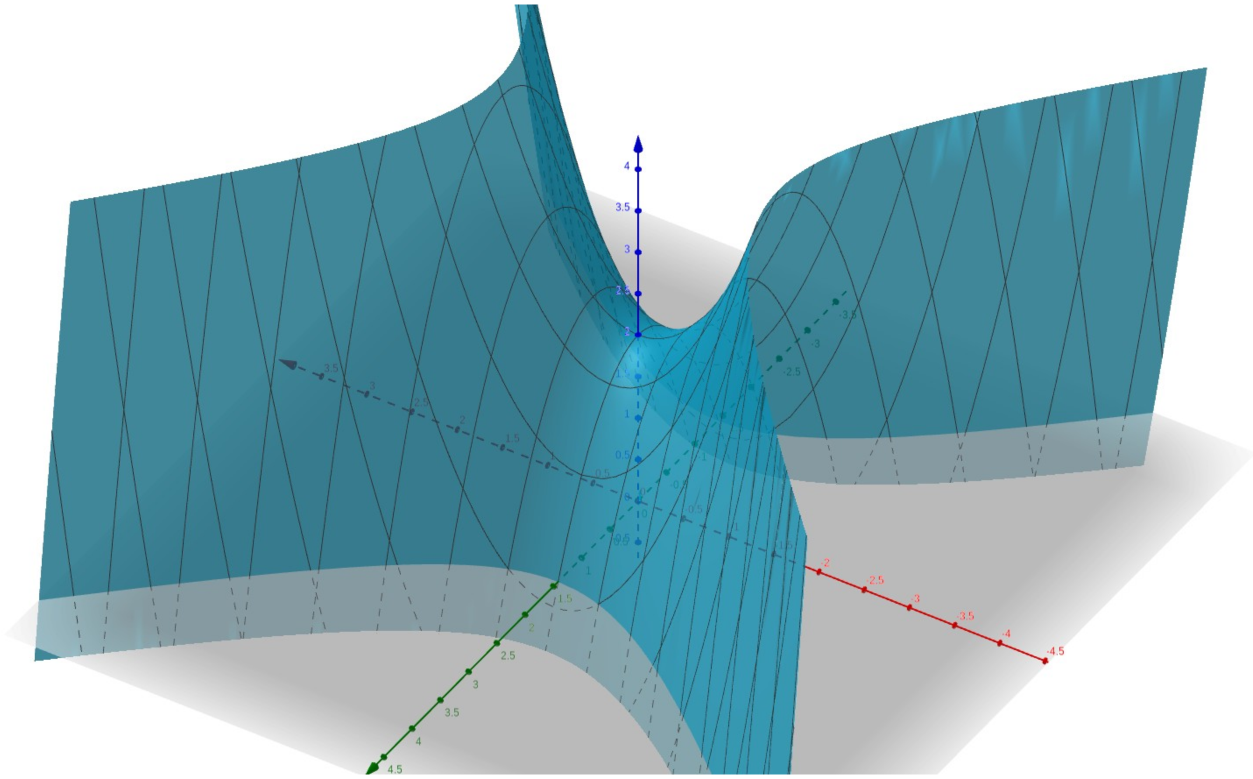
  ($\langle h, H_f h \rangle = 0$ for some $h \in \mathbb{R}^n$)

- $f(x,y) = y^3 - 3x^2 y + 2 \implies H_f(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \implies$ degenerate point

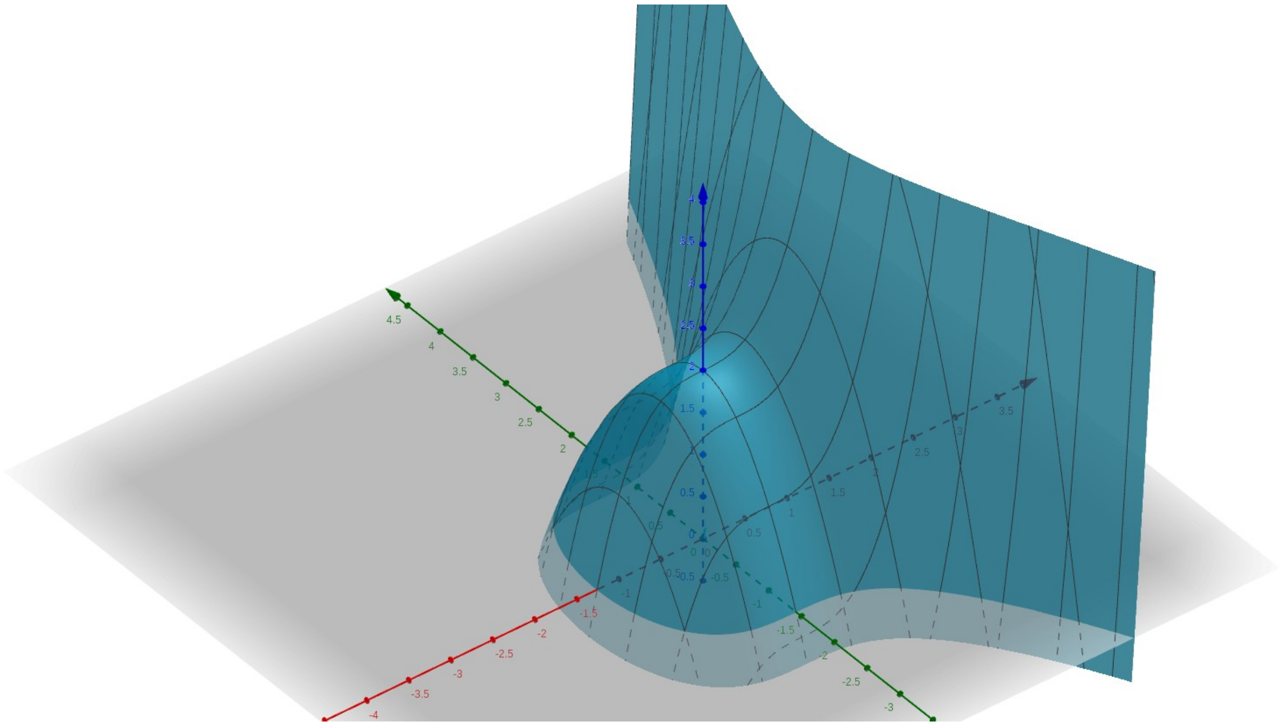
  ("monkey saddle", see pictures)

The following pictures were generated with https://www.geogebra.org/3d.

$$f(x,y)=x^2-y^2+2$$



$$f(x,y)=x^3-y^2+2$$



$$f(x,y)=y^3-3x^2y+2$$

("Monkey Saddle")