2.3 The Inverse and Implicit Function Theorems

Question: Under which conditions is $\underbrace{f: U \to \mathbb{R}^n \quad (U \subset \mathbb{R}^n \text{ open})}$ invertible?

Here f goes from a subset of $\mathbb{R}^n$ into a subset of $\mathbb{R}^n$.

And: If f is invertible and differentiable, is then $\underbrace{f^{-1}}_{\text{the inverse of } f}$ also differentiable?

Reminder:

- $f: X \to Y$ is called $\boxed{\text{injective}}$ (or "one-to-one") if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.
  (In other words: Given $y \in Y$, then $f(x) = y$ for at most one $x \in X$.)

  Ex.: $f: \mathbb{R} \to \mathbb{R}, x \mapsto e^x$ is injective, but $f: \mathbb{R} \to \mathbb{R}, x \mapsto x^2$ is not.

- $f: X \to Y$ is called $\boxed{\text{surjective}}$ (or "onto") if $\forall y \in Y$ there is an $x \in X$ s.t. $f(x) = y$.

  Ex.: $f: \mathbb{R} \to \mathbb{R}, x \mapsto e^x$ is not surjective, but $f: \mathbb{R} \to (0, \infty), x \mapsto e^x$ is.

- $f: X \to Y$ is called bijective if it is injective and surjective.
  If $f: X \to Y$ is bijective, then it has an inverse $f^{-1}: Y \to X$
  (i.e., $f(f^{-1}(y)) = y \;\; \forall y \in Y$, or $f^{-1}(f(x)) = x \;\; \forall x \in X$.)

From Analysis and Calculus we know the case $n=1$:

If f is continuously differentiable and $f'(p)\neq 0$, then f is invertible in a neighborhood of p, f^{-1} is continuously differentiable, and $\underbrace{(f^{-1})'(f(p)) = \frac{1}{f'(p)}}$.

If $f(p)=q$, then $(f^{-1})'(q) = \frac{1}{f'(f^{-1}(q))}$

For general n, we have the following theorem.

Theorem (Inverse Function Theorem):

Let $U\subset\mathbb{R}^n$ be open, $f: U\to\mathbb{R}^n$ be $C^1(U)$, and let $Df|_p$ be invertible for some $p\in U$.

Then:

a) There are open neighborhoods V of p and W of $q:=f(p)$ s.t. $f|_V: V\to W$ is bijective (i.e., $f|_V$ has an inverse).

b) The inverse $(f|_V)^{-1}$ is $C^1(V)$.

Note:

- $Df|_p$ invertible $\iff$ The Jacobian matrix $J_{ij}(p) = \frac{\partial f_i(p)}{\partial x_j}$ is invertible.
- Using the chain rule we find: $\underset{\substack{\uparrow \\ \text{identity} \\ \text{on } \mathbb{R}^n}}{1} = \underbrace{D(f^{-1}\circ f)|_p}_{\substack{\text{derivative of} \\ f^{-1}(f(x))=x}} = Df^{-1}|_{f(p)}\, Df|_p$ ← chain rule

$$\Rightarrow \boxed{Df^{-1}|_{f(p)} = (Df|_p)^{-1}}$$

- The inverse fct. thm. implies: The system of equations $f_i(x_1,\dots,x_n)=y_i$, $i=1,\dots,n$ can be solved for $x_1,\dots,x_n$ in terms of $y_1,\dots,y_n$, if x and y are in small enough neighborhoods of p and q.

- If $f: V \to W$ is C^k, and f^{-1} exists and is C^k then f is called a C^k diffeomorphism.
- If any $p \in V$ has a neighborhood $\hat{V}$ s.t. $f|_{\hat{V}} : \hat{V} \to f(\hat{V})$ is a diffeomorphism, then f is called a local diffeomorphism.

For the proof, we use an important theorem.

First, on a metric space (X,d), a map $f: X \to X$ is called a contraction if there is $0 \le c < 1$ s.t. $d(f(x), f(y)) \le c\, d(x,y)$.

A point $x^* \in X$ is called fixed point if $f(x^*) = x^*$.

Note: Suppose f is a contraction and it has two fixed points: $f(x_1) = x_1$, $f(x_2) = x_2$.

Then $\quad d(x_1, x_2) = d(f(x_1), f(x_2)) \le c\, d(x_1, x_2)$ with $0 < c \le 1$, which implies $d(x_1, x_2) = 0$, i.e., $x_1 = x_2$.

So if a contraction has a fixed point, then it is unique.

Moreover:

Banach Fixed-Point Theorem (or: Contraction Mapping Principle):

If X is a complete metric space, then any contraction $f: X \to X$ has a unique fixed point.

Proof: See homework 4.

(Define $x_{n+1} := f(x_n)$ $\forall n$ and show that (x_n) is Cauchy $\Rightarrow$ limit x^* exist since X is complete $\Rightarrow$ $f(x^*) = f(\lim_{n\to\infty}(x_n)) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x^*$.) $\square$

f continuous. Why?

Note: This proof gives us an explicit way to construct the fixed point:

It is the limit of the sequence $x_{n+1} = f(x_n)$.

(I.e., choose some x_0, then $x_1 = f(x_0)$, $x_2 = f(x_1) = f(f(x_0))$, i.e., $x_n = f^{\circ n}(x_0)$.)

Example: Newton's method for finding zeroes of $f(x)$.

(Extra example not covered in the in-person class)

We guess/hope that the iteration $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} =: F(x_n)$ converges to a zero of f. Suppose $f(x^*) = 0$, $f'(x^*) \neq 0$. Then $F(x^*) = x^*$, i.e., x^* is a fixed point of the map F.

With the Banach Fixed-Point Theorem we could now find sufficient conditions for Newton's method to converge by constructing a suitable complete metric space X on which F maps $X \to X$ and is a contraction.

E.g.: For $f(x) = x^2 - 3$, we have $F(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^2-3}{2x} = \frac{1}{2}\left(x + \frac{3}{x}\right)$.

Here $F: [\sqrt{3}, \infty) \to [\sqrt{3}, \infty)$, i.e., we can choose $X = [\sqrt{3}, \infty)$ (which is closed, so X with the standard metric (absolute value) is indeed complete).

Is F a contraction on X?

$$d(F(x), F(y)) = |F(x) - F(y)| = \frac{1}{2}\left|\left(x + \frac{3}{x}\right) - \left(y + \frac{3}{y}\right)\right|$$

$$= \frac{1}{2}\left|x - y + 3\underbrace{\left(\frac{1}{x} - \frac{1}{y}\right)}_{= \frac{y-x}{xy}}\right| = \frac{1}{2}\left|(x-y)\left(1 - \frac{3}{xy}\right)\right|$$

$$\leq \frac{1}{2}|x-y|\underbrace{\left|1 - \frac{3}{xy}\right|}_{\leq 1} \leq \frac{1}{2}|x-y|$$

, so yes, F is a contraction.

So, by Banach's fixed point thm., $x_{n+1} = F(x_n)$ converges to the unique fixed point $x^* = \sqrt{3}$ for any initial $x_0 \in [\sqrt{3}, \infty)$.

(In fact, for $x_0 \in (0, \sqrt{3})$, we have $x_1 = F(x_0) = \frac{1}{2}\left(x_0 + \frac{3}{x_0}\right) > \sqrt{3}$, so we could use any $x_0 > 0$ as initial point for the iteration.)