Last time we discussed:

Theorem (Inverse Function Theorem):

Let $U \subset \mathbb{R}^n$ be open, $f: U \to \mathbb{R}^n$ be $C^1(U)$, and let $Df|_p$ be invertible for some $p \in U$.

Then:

a) There are open neighborhoods V of p and W of $q := f(p)$ s.t. $f|_V : V \to W$ is bijective (i.e., $f|_V$ has an inverse).

b) The inverse $(f|_V)^{-1}$ is $C^1(V)$.

Proof of Inverse Function Theorem:

a) Idea:
- Injectivity: We conclude $f(x) = y$ for at most one x by constructing a contraction s.t. x = fixed point (if it exists, it is unique).
- Surjectivity: We construct a complete metric space X s.t. we can apply the Banach fixed point theorem.

- Injectivity:

Let us call $A := Df|_p$, and choose $\lambda := \frac{1}{2\|A^{-1}\|}$. (This specific choice will become clear later.)

Since Df is continuous at p, $\exists$ open ball $\tilde{U} \subset U$ centered at p s.t.

$$\|Df|_x - Df|_p\| \leq \lambda \quad \forall x \in \tilde{U}.$$

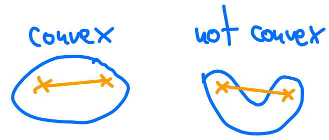
For any (fixed) $y \in \mathbb{R}^n$, we def. $\varphi_y(x) := x + A^{-1}(y - f(x))$,

because then: $f(x) = y \iff \varphi_y(x) = x$.

Is φ_y a contraction? If yes, then $\underbrace{\exists \text{ at most one fixed point}}_{\text{as we showed last time}}$, i.e., $f(x) = y$ for at most one x, i.e., $f|_{\tilde{U}}$ is injective.

We try to bound $\|\varphi_y(x_1) - \varphi_y(x_2)\|$ by bounding the derivative $D\varphi_y|_x$ because of the following

All points on a straight line between any $x, y \in U$ are in U.



<u>Lemma:</u> Let $f: U \to \mathbb{R}^n$, $U \subset \mathbb{R}^n$ open and convex. If f is differentiable and $\exists M > 0$ s.t. $\|f'(x)\| \leq M \;\; \forall x \in U$, then $\|f(x_1) - f(x_2)\| \leq M \|x_1 - x_2\| \;\; \forall x_1, x_2 \in U$.

<u>Proof:</u> Define curve $\gamma: [0,1] \to \mathbb{R}^n$, $t \mapsto t x_1 + (1-t) x_2$. Then γ is in U because U is convex. If $g(t) := f(\gamma(t))$, then

$$f(x_1) - f(x_2) = g(1) - g(0) = \int_0^1 g'(t)\,dt, \text{ where } g'(t) \overset{\text{chain rule}}{=} f'(\gamma(t))\,\gamma'(t) = f'(\gamma(t))\,(x_1 - x_2).$$

$$\Rightarrow \|f(x_1) - f(x_2)\| \leq \int_0^1 \|g'(t)\|\,dt = \int_0^1 \underbrace{\|f'(\gamma(t))\|}_{\leq M} \|x_1 - x_2\|\,dt \leq M \|x_1 - x_2\|. \qquad \square$$

Back to our map φ_y.

We compute: $D\varphi_y|_x = 1 - A^{-1} Df|_x = A^{-1}(A - Df|_x)$.

Then for $x \in \tilde{U}$ we have $\| D\varphi_y|_x \| \leq \| A^{-1}(A - Df|_x)\|$

$$\leq \|A^{-1}\| \underbrace{\|Df|_p - Df|_x\|}_{\leq \lambda = \frac{1}{2\|A^{-1}\|}}$$

$$\leq \frac{1}{2}$$

Thus, by the lemma, we have $\|\varphi_y(x_1) - \varphi_y(x_2)\| \leq \frac{1}{2}\|x_1 - x_2\| \quad \forall x_1, x_2 \in \tilde{U}$, i.e., φ_y is a contraction.

- Surjectivity: With a similar argument, we can show that one can choose a closed ball

$$\overline{B_r(p)} = \{x \in \mathbb{R}^n : \|x-p\| \leq r\} \subset U$$ such that $\varphi_y : \overline{B_r(p)} \to \overline{B_r(p)}$. Then existence of a fixed point follows from Banach's fixed point theorem.

b) First, we use: If $Df|_p$ has an inverse, then also $Df|_x$ for $\|x-p\|$ small enough has an inverse. (The set of invertible linear maps on $\mathbb{R}^n$ is open; see Rudin for a proof.)

Then differentiability of f^{-1} can be proven by showing

$$\frac{f^{-1}|_{y+k} - f^{-1}_y - (Df|_x)^{-1}k}{\|k\|} \xrightarrow{k \to 0} 0. \qquad \text{(See Rudin for the details.)}$$

Continuity of Df^{-1} follows from continuity of $Df|_x$. $\square$

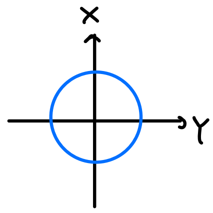
Next: let $f: \mathbb{R}^{n+m} \to \mathbb{R}^n$. Q.: Under which conditions can we solve $f(x,y)=0$ for $x \in \mathbb{R}^n$ in terms of $y \in \mathbb{R}^m$?

In other words: In the system of equations

$$\begin{aligned} f_1(x_1,\dots,x_n,y_1,\dots,y_m) &= 0 \\ &\vdots \\ f_n(x_1,\dots,x_n,y_1,\dots,y_m) &= 0, \end{aligned}$$

can we solve for $x_1(y_1,\dots,y_m), \dots, x_n(y_1,\dots,y_m)$, at least locally?

<u>Ex.:</u> $f(x,y) = x^2 + y^2 - 1$, $x, y \in \mathbb{R}$.



$\Rightarrow$ $f(x,y)=0$ has two local solutions $x_\pm(y) = \pm\sqrt{1-y^2}$.

More precisely: Solution possible in an open neighborhood except when $x=0$ $(y=\pm 1)$.

At $x=0$, we have $\left.\frac{\partial f}{\partial x}\right|_{x=0} = \left.2x\right|_{x=0} = 0$, i.e., $\left.\frac{\partial f}{\partial x}\right|_{x=0}$ not invertible.

$\Rightarrow$ It seems we require $\frac{\partial f}{\partial x}$ to be invertible

This is generalized in the Implicit Function Theorem, which we will discuss next time.