Theorem (Implicit Function Theorem):

Let $U \subset \mathbb{R}^{n+m}$ be open, $f: U \to \mathbb{R}^n$ be $C^1(U)$, and $f(p,q)=0$ for some $(p,q)\in U$.

We assume that $\frac{\partial f}{\partial x}(p,q) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}\Big|_{(p,q)}$ is invertible.

Then there are open sets $V \subset \mathbb{R}^{n+m}$ and $W \subset \mathbb{R}^m$ with $(p,q)\in V$, $q \in W$ s.t. to every $y \in W$ corresponds a unique x s.t. $(x,y)\in V$ and $f(x,y)=0$. If this $x := g(y)$, then $g: W \to \mathbb{R}^n$ is C^1, $g(q)=p$, $f(g(y),y)=0$, and $Dg|_q = -\left(\frac{\partial f}{\partial x}\right)^{-1}\Big|_{(p,q)} \frac{\partial f}{\partial y}\Big|_{(p,q)}$.

In our example above: $f(x,y)=x^2+y^2-1$, $x\neq 0$.

$\Rightarrow g(y) := \sqrt{1-y^2}$ for $y>0$ $\Rightarrow f(g(y),y)=0$, and $\frac{\partial g}{\partial y} = \frac{-\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial x}} = \frac{-2y}{2x} = \frac{-y}{\sqrt{1-y^2}}$. ✓

Idea of proof:

We define $F: U \to \mathbb{R}^{n+m}$ by $F(x,y) := \begin{pmatrix} f(x,y) \\ y \end{pmatrix}$.

Then $F(p,q) = (0,q)$ and $DF|_{(p,q)} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \underbrace{0}_{\frac{\partial y}{\partial x}} & \underbrace{\mathbb{1}}_{\frac{\partial y}{\partial y}} \end{pmatrix}$.

$\Rightarrow \det DF|_{(p,q)} = \det \frac{\partial f}{\partial x}\Big|_{(p,q)} \neq 0$, so $DF|_{(p,q)}$ is invertible and we can use the inverse fct. thm. to invert $\mathcal{F}$ in neighborhoods V of (p,q) and W of $(0,q)$.

If $(0,y) \in W$, then $(0,y) = \mathcal{F}(x,y)$ for some $(x,y) \in V$, i.e., $f(x,y) = 0$.

Next: • x is unique since $\mathcal{F}$ is bijective.

• The fct. g is C^1 since $\mathcal{F}^{-1}$ is C^1.

• Derivative formula follows from chain rule.

see Rudin for details

$\square$

An application:

A surface $M \subset \mathbb{R}^3$ can be defined via $F(x,y,z) = 0$, $F: U \to \mathbb{R}^3$, i.e., $M = \{(x,y,z) \in U : F(x,y,z) = 0\}$. Then the implicit fct. thm. tells us that if $F \in C^1(U)$ and $\frac{\partial F}{\partial z} \neq 0$, then locally the surface can be defined via the explicit equation $z = \phi(x,y)$.

Surfaces are special cases of manifolds, a concept that will be introduced in Analysis III.

Note: In our chapter on many variable differentiation, we skipped a discussion on Lagrange multipliers. This is discussed in Calculus and Linear Algebra II.

3. Integrals

Generally, there are 3 ways to integrate in many variables:

- Successive 1-dim. Riemann integrals: $\int\limits_{[a,b]\times[c,d]} f(x_1,x_2)\,d^2x := \int\limits_c^d \left(\int\limits_a^b f(x_1,x_2)\,dx_1\right)dx_2.$

  Then an important question is: Is $\int\left(\int f(x_1,x_2)dx_1\right)dx_2 = \int\left(\int f(x_1,x_2)dx_2\right)dx_1$?

- Re-define the Riemann integral in n-dim., using partitions of $\mathbb{R}^n$.

  Question: Is it equal to successive 1-dim. integration?

- Lebesgue integral: see Analysis III.

We start here with considering partial integrals $F(y) = \int\limits_a^b f(x,y)\,dx$.