3.1 Partial Integrals

We first consider partial integrals, i.e., $F(y) := \int_a^b f(x,y)\,dx$.

Here, $I = [a,b] \times [\alpha,\beta] = I_1 \times I_2$, $f: I \to \mathbb{R}$, $\underbrace{f(\cdot,y)}$ is integrable for every $y \in I_2$, $F: I_2 \to \mathbb{R}$.

f as a fct. of the variable in the first slot, for fixed y

A key property to understand partial integrals is uniform continuity.

<u>Definition:</u> Let X, Y be metric spaces. Then $f: X \to Y$ is called uniformly continuous if $\underbrace{\forall \varepsilon > 0 \; \exists \delta > 0 \text{ s.t. } \forall x, x' \in X}_{\text{one } \varepsilon \text{ works } \forall x,x' \in X}$ with $d(x,x') < \delta$ we have $d(f(x), f(x')) < \varepsilon$.

One ε works $\forall x, x' \in X$ (compare with uniform convergence)

We will use the following result (applied to the compact sets I, I_1, I_2):

<u>Theorem:</u> If K is compact and $f: K \to Y$ continuous, then f is uniformly continuous.

<u>Proof</u> (using that K compact $\Leftrightarrow$ K sequentially compact, i.e., every sequence has a converging subsequence): Assume f not uniformly continuous, i.e., $\exists \varepsilon > 0$ s.t. for $\delta = \frac{1}{n}$ there are $(x_n)_n$ and $(x'_n)_n$ with $\underbrace{d(x_n, x'_n) < \tfrac{1}{n}}_{(*)}$ but $\underbrace{d(f(x_n), f(x'_n)) \geq \varepsilon}_{(**)}$.

Then: K compact $\Rightarrow$ $(x_n), (x_n')$ have converging subsequences $(x_{n_j}), (x_{n_j}')$:

$x_{n_j} \to x \in K$, $x_{n_j}' \to x' \in K$ and $x = x'$ due to $(*)$.

Since f continuous, $f(x_{n_j}) \to f(x)$ and $f(x_{n_j}') \to f(x)$, which contradicts $(**)$. $\square$

Back to the partial integral $F(y) := \int_a^b f(x,y)\,dx$.

First, we aim at proving $\frac{dF(y)}{dy} = \int_a^b \frac{\partial f(x,y)}{\partial y}\,dx$. A small intermediate result is:

Theorem: If $\underbrace{f \in C(I)}_{f \text{ is continuous on } I = I_1 \times I_2}$, then $F \in C(I_2)$.

Proof: Let $\varepsilon > 0$. $f \in C(I) \Rightarrow f$ uniformly continuous on $I \Rightarrow \exists\, \delta > 0$ s.t.

$\forall x, y, y'$ with $\underbrace{|y - y'|}_{= |(x,y) - (x,y')|} < \delta$: $|f(x,y) - f(x,y')| < \frac{\varepsilon}{b-a}$.

Then $|F(y) - F(y')| = \left| \int_a^b (f(x,y) - f(x,y'))\,dx \right| \leq \int_a^b \underbrace{|f(x,y) - f(x,y')|}_{\leq \frac{\varepsilon}{b-a}}\,dx \leq (b-a) \frac{\varepsilon}{b-a} = \varepsilon.$

$\square$

Then the following holds:

Theorem (Leibnitz' rule I):

If $f \in C(I)$ and $\frac{\partial f}{\partial y} \in C(I)$, then $F \in C^1(I_2)$ and $\underbrace{\frac{dF}{dy}(y) = \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx}_{(*)}$.

Proof: Let $\varepsilon > 0$. Since $\frac{\partial f}{\partial y} \in C(I)$, it is uniformly continuous.

Therefore, $\exists\, \delta > 0$ s.t. $\forall x \in I_1, y, y' \in I_2$ with $|y - y'| < \delta$:

$$\left| \frac{\partial f}{\partial y}(x,y) - \frac{\partial f}{\partial y}(x,y') \right| < \frac{\varepsilon}{b-a}.$$

Then, for $|h| < \delta$ $(y + h \in I_2)$:

$$\Rightarrow \left| \frac{F(y+h)-F(y)}{h} - \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx \right| \le \left| \int_a^b \Bigg(\underbrace{\frac{f(x,y+h)-f(x,y)}{h}}_{=\frac{\partial f}{\partial y}(x,y+\theta h),\ 0<\theta<1,\ \text{by the mean-value thm.}} - \frac{\partial f}{\partial y}(x,y) \Bigg) dx \right|$$

$$\le (b-a)\frac{\varepsilon}{b-a} = \varepsilon.$$

$\Rightarrow F$ differentiable and $(*)$ holds. With previous thm. applied to $\frac{\partial f}{\partial y}$, F' is continuous (i.e., $F \in C^1(I_2)$.) $\square$

What about indefinite integrals?

Theorem (Leibnitz' rule II): Let $I_1 = [a,\infty)$, $I_2 = [\alpha,\beta]$, $I = I_1 \times I_2$, f and $\frac{\partial f}{\partial y} \in C(I)$.

Assume: (i) $F(y) = \int_a^\infty f(x,y)\,dx$ converges $\forall y \in I_2$.

(ii) $\int_a^\infty \frac{\partial f}{\partial y}(x,y)\,dx$ converges absolutely and uniformly on I_2.

↳ i.e., $g_n(y) := \int_a^n \frac{\partial f}{\partial y}(x,y)\,dx$ conv. abs. and uniformly

Then $F \in C^1(I_2)$ and $F'(y) = \int_a^\infty \frac{\partial f}{\partial y}(x,y)\,dx$.

Proof: Analogous to the previous proof, we find

$$\left| \frac{F(y+h)-F(y)}{h} - \int_a^\infty \frac{\partial f}{\partial y}(x,y)\,dx \right| \le \int_a^\infty \left| \frac{\partial f}{\partial y}(x,y+\theta h) - \frac{\partial f}{\partial y}(x,y) \right| dx$$

$$\le \underbrace{\int_a^b \left| \frac{\partial f}{\partial y}(x,y+\theta h) - \frac{\partial f}{\partial y}(x,y) \right| dx}_{\le \frac{\varepsilon}{3}} + \underbrace{\int_b^\infty \left| \frac{\partial f}{\partial y}(x,y+\theta h) \right| dx}_{\le \frac{\varepsilon}{3}} + \underbrace{\int_b^\infty \left| \frac{\partial f}{\partial y}(x,y) \right| dx}_{\le \frac{\varepsilon}{3}}$$

bc. as in previous proof: we can chose $\delta = \frac{\varepsilon}{3(b-a)}$ due to uniform continuity

by uniform convergence ($\forall \varepsilon > 0\ \exists b > a$ s.t. $\int_b^\infty \left| \frac{\partial f}{\partial y} \right| dx < \frac{\varepsilon}{3}$, uniformly $\forall y \in I_2$)

Continuity of $F'(y) = \int_a^\infty \frac{\partial f}{\partial y}(x,y)\,dx$ follows as before and with the same argument of splitting $\int_a^\infty \ldots = \int_a^b \ldots$. $\square$

Example: $f(x,y)=e^{-xy}\frac{\sin x}{x}$ on $I=[0,\infty)\times[\alpha,\beta]$, $0<\alpha<\beta$ (and $f(0,y):=0$).

Here, $\frac{\partial f}{\partial y}(x,y)=-e^{-xy}\sin x$.

We have $y\geq\alpha>0$ (so $e^{-xy}\leq e^{-x\alpha}$) and $\left|\frac{\sin x}{x}\right|\leq 1$ (and $|\sin x|\leq 1$), so conditions (i) and (ii) from the theorem hold.

We find $F'(y)=\int_0^\infty \frac{\partial f}{\partial y}(x,y)\,dx=-\int_0^\infty e^{-xy}\sin x\,dx$

integration by parts

$$= \underbrace{-e^{-xy}(-\cos x)\Big|_0^\infty}_{=0-1}+(-y)\int_0^\infty e^{-xy}(-\cos x)\,dx$$

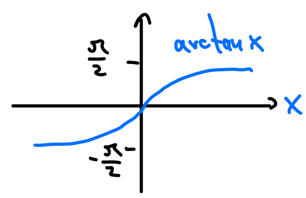
$$=-1+y\left[\underbrace{e^{-xy}\sin x\Big|_0^\infty}_{=0-0}-(-y)\int_0^\infty e^{-xy}\sin x\,dx\right]$$

$$=-1+y^2\underbrace{\int_0^\infty e^{-xy}\sin x\,dx}_{=-F'(y)}$$

$$\Rightarrow F'(y)=-1-y^2F'(y)\Rightarrow F'(y)=\frac{-1}{1+y^2}$$

[check this integration]

$$\Rightarrow F(\beta)-F(\alpha)=\int_\alpha^\beta F'(y)\,dy=\arctan\alpha-\arctan\beta.$$



Note that $|F(\beta)|\leq\int_0^\infty|e^{-x\beta}|\,dx\leq\frac{1}{\beta}\xrightarrow{\beta\to\infty}0$, so for $\beta\to\infty$ we get

$$0-F(\alpha)=\arctan\alpha-\frac{\pi}{2}\Rightarrow F(\alpha)=\frac{\pi}{2}-\arctan\alpha,$$

where $F(\alpha)=\int_0^\infty e^{-x\alpha}\frac{\sin x}{x}\,dx$.

It can be shown that F is indeed continuous (from the right), so we have computed the Dirichlet integral $F(0)=\int_0^\infty\frac{\sin x}{x}\,dx=\frac{\pi}{2}$. ("Feynman's trick")

[check that this is indeed Riemann integrable]