Advanced Calculus and Methods of Mathematical Physics — Session 16

Prof. Sören Petrat, Jacobs University, Spring 2022 — Mar. 25, 2022

For the next generalization of Leibnitz' rule, the integration boundaries can be variable:

$\boxed{\text{\underline{Theorem (Leibnitz' rule III):}}}$ For $I = [a,b] \times [\alpha,\beta] = I_1 \times I_2$, let f and $\frac{\partial f}{\partial y} \in C(I)$, and $\phi, \psi : I_2 \to I_1$ be $C^1(I_2)$. Let
$$H(y) := \int_{\phi(y)}^{\psi(y)} f(x,y)\,dx.$$
Then $H \in C^1(I_2)$ and
$$H'(y) = \int_{\phi(y)}^{\psi(y)} \frac{\partial f}{\partial x}(x,y)\,dx + f(\psi(y),y)\,\psi'(y) - f(\phi(y),y)\,\phi'(y).$$

<u>Proof:</u> Define $F(y,u,v) := \int_u^v f(x,y)\,dx$ and $G(y) = (y, \phi(y), \psi(y))$, then $\underbrace{H = F \circ G}_{H(y) = F(G(y))}$.

For fixed u and v, F satisfies the conditions of Leibnitz' rule I, so
$$\frac{\partial F}{\partial y} = \int_u^v \frac{\partial f}{\partial y}(x,y)\,dx.$$

Then the chain rule gives
$$H'(y) = (dF \circ G)\,G' = \left(\int_u^v \frac{\partial f}{\partial y}(x,y)\,dx, -f, f \right) \circ G \begin{pmatrix} 1 \\ \phi'(y) \\ \psi'(y) \end{pmatrix}$$
$$= \int_{\phi(y)}^{\psi(y)} \frac{\partial f}{\partial y}(x,y)\,dx - f(\phi(y),y)\,\phi'(y) + f(\psi(y),y)\,\psi'(y). \quad \square$$

With these three theorems, we have a good understanding of how to exchange integration and differentiation. (Note: Much nicer conditions hold for the Lebesgue integral → Analysis III.)

Next: Does the order of integration matter?

Theorem: let $f \in C(I)$, $I = [a,b] \times [\alpha, \beta]$. Then

$$\int_\alpha^\beta \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_\alpha^\beta f(x,y)\,dy\,dx.$$

Proof: Idea: estimate integrals on small rectangles and then use uniform continuity.

let $\varepsilon > 0$. Since f uniformly continuous:

$$\exists \delta > 0 \text{ s.t. if } d((x,y),(x',y')) < \delta, \text{ then } |f(x,y) - f(x',y')| < \frac{\varepsilon}{(b-a)(\beta-\alpha)}. \quad (*)$$

Now we partition the x and y axis: Def. $a = x_0 < x_1 < \dots < x_n = b$ and $\alpha = y_0 < y_1 < \dots < y_m = \beta$ such that $I_{ij} := [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ has diameter smaller δ $\forall i,j$.

We def. $m_{ij} = \min_{(x,y) \in I_{ij}} f(x,y)$ and $M_{ij} = \max_{(x,y) \in I_{ij}} f(x,y)$.

If $A_{ij} = \text{area}(I_{ij}) = (x_i - x_{i-1})(y_j - y_{j-1})$, then

$$m_{ij} A_{ij} \leq \int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} f(x,y)\,dy\,dx \leq M_{ij} A_{ij}.$$

Summing up yields:

$$\sum_{i,j} m_{ij} A_{ij} \leq \int_a^b \int_\alpha^\beta f(x,y)\,dy\,dx \leq \sum_{i,j} M_{ij} A_{ij}. \quad (**)$$

This argument works just as well for the other order of integration, i.e.,

$$\sum_{i,j} m_{ij} A_{ij} \leq \int_\alpha^\beta \int_a^b f(x,y)\,dx\,dy \leq \sum_{i,j} M_{ij} A_{ij}. \quad (**)'$$

Since $\left| \sum_{i,j} m_{ij} A_{ij} - \sum_{i,j} M_{ij} A_{ij} \right| \leq \sum_{i,j} A_{ij} \underbrace{|m_{ij} - M_{ij}|}_{\leq \frac{\varepsilon}{(b-a)(\beta-\alpha)} \text{ by } (*)} \leq \varepsilon,$

the inequalities $(**)$ and $(**)'$ yield

$$\left| \int_a^b \int_\alpha^\beta f(x,y)\,dy\,dx - \int_\alpha^\beta \int_a^b f(x,y)\,dx\,dy \right| \leq \varepsilon.$$ (This holds $\forall \varepsilon > 0$, i.e., the left-hand side $= 0$.) $\square$