Advanced Calculus and Methods of Mathematical Physics — Session 17

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Last time we proved: For $f \in C(I)$, we have: $\int_\alpha^\beta \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_\alpha^\beta f(x,y)\,dy\,dx$.

Examples:

- $f(x,y) = x^y$ on $I = [0,1] \times [\alpha,\beta]$, with $0 < \alpha < \beta$. We have:

$$\int_\alpha^\beta x^y\,dy = \int_\alpha^\beta e^{y \ln x}\,dy = \frac{1}{\ln x} e^{y\ln x} \Big|_{y=\alpha}^{y=\beta} = \frac{x^\beta - x^\alpha}{\ln x},$$

$\exp \ln x^y = e^{y \ln x}$

$$\int_0^1 x^y\,dx = \frac{x^{y+1}}{y+1}\Big|_{x=0}^{x=1} = \frac{1}{y+1}.$$

Thus, $\int_0^1 \frac{x^\beta - x^\alpha}{\ln x}\,dx = \int_\alpha^\beta \frac{1}{y+1}\,dy = \ln(y+1)\Big|_\alpha^\beta = \ln(1+\beta) - \ln(1+\alpha) = \ln\frac{1+\beta}{1+\alpha}$.

- $f(x,y) = \frac{x^2 - y^2}{(x^2+y^2)^2}$ on $I = [0,1] \times [\alpha,1]$. See Problem 4, Homework 7.

  Note: $f(x,y)$ is <u>not</u> continuous on $[0,1]\times[0,1]$. In fact, we show in the homework that

$$\int_0^1 \int_0^1 f(x,y)\,dx\,dy = -\frac{\pi}{4} \neq \int_0^1 \int_0^1 f(x,y)\,dy\,dx = \frac{\pi}{4}.$$

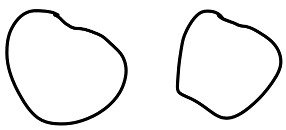
Next: How do we integrate over more general sets in $\mathbb{R}^2$ (or, more generally, $\mathbb{R}^n$)?

3.2 The Riemann Integral in $\mathbb{R}^n$

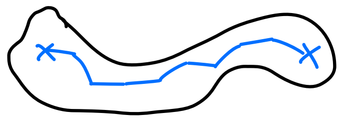
Strategy: We define a Riemann integral on $\mathbb{R}^n$ "from scratch", valid for integrating over a large set of $D \subset \mathbb{R}^n$. Then we make the connection to repeated 1-dim. Riemann integrals.

Definition: A domain in $\mathbb{R}^n$ is a non-empty connected open set.

Note: $A \subset \mathbb{R}^n$ connected means that any two points in A can be connected by a polygonal path.
(Note: there is a more general topological def. of connectedness.)



not connected



connected

Furthermore: x is a boundary point of $A \subset \mathbb{R}^n$ if every open neighborhood of x contains a point in A and in A^c. ($A^c = \mathbb{R}^n \setminus A$ is the complement of A.)

We denote: • $\partial A = \{\text{all boundary points of } A\}$ the boundary of A (e.g., $\partial\{\|x\|<r\} = \{\|x\|=r\}$)

• $\overline{A} = A \cup \partial A$ the closure of A

• $\text{int}(A) = \left(\overline{A^c}\right)^c$ the interior of A

↳ e.g., $\text{int}\{x\in\mathbb{R}^n : \|x\| \leq r\} := \overline{\{x\in\mathbb{R}^n : \|x\|>r\}}^c = \{x\in\mathbb{R}^n : \|x\|\geq r\}^c = \{x\in\mathbb{R}^n : \|x\|<r\}$

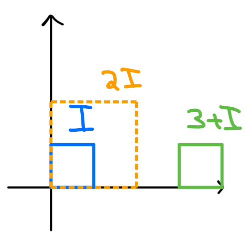
(Note: $\partial A = \overline{A} \setminus \text{int}(A)$.)

We aim at defining the content ("volume") $S(A)$ for $A \subset \mathbb{R}^n$.
↓
also called "Jordan content" or "Jordan measure"

We define:

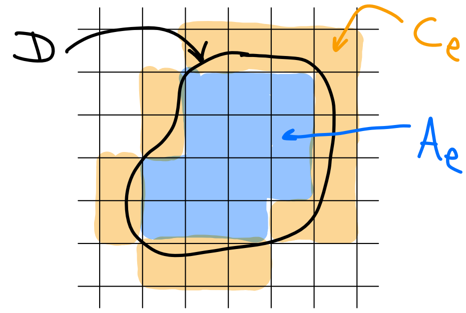
• The unit cell $I = [0,1]^n$ has content $S(I) = 1$.

• Let $I_k = k + I$ be the unit cell translated by k, ϱI the unit cell dilated by ϱ ($\varrho > 0$).
Then $S(k + \varrho A) = \varrho^n S(A)$.



Then, for $e>0$, we divide $\mathbb{R}^n$ into cells, $\mathbb{R}^n = \bigcup_{k\in\mathbb{Z}^n} e I_k$, and we def. for $D\subset\mathbb{R}^n$:

- $A_e = \bigcup\{eI_k \text{ inside } D\}$
- $C_e = \bigcup\{eI_k \text{ hits the boundary of } D\}$



Definition: Given a domain $D\subset\mathbb{R}^n$, we say D (and $\overline{D}$) "has content" or "is Jordan measurable" if $\lim_{e\to 0} S(A_e)$ and $\lim_{e\to 0} S(A_e\cup C_e)$ exist and are equal.

Example: $D=[0,1]\cap\mathbb{Q}$.

Since $\partial\mathbb{Q}=\mathbb{R}$ and $\operatorname{int}(\mathbb{Q})=\emptyset$, we have $S(A_e)=0$, $S(A_e\cup C_e)=S(C_e)=1$, so D is not Jordan measurable. (Note: D will turn out to be Lebesgue measurable.)

Next: Partitions, Riemann sums $\to$ Riemann integrability

Definition: A partition of $\overline{D}$ is a family $T=\{\overline{D}_j, j=1,\dots,k\}$ such that

a) $\overline{D}_j\subset D$ are subdomains with content,

b) $\{D_j\}$ disjoint,

c) $\overline{D}=\bigcup_{j=1}^{k}\overline{D}_j$.

We call $\lambda(T)=$ the maximal diameter of all D_j's the "parameter" or "mesh" of T.

Definition: Let $f: \bar{D} \to \mathbb{R}$ be bounded ($\bar{D} \subset \mathbb{R}^n$ a closed domain). A **Riemann sum** for f is a sum

$$\sigma(f, T, x_1, \dots, x_k) = \sum_{j=1}^{k} f(x_j) \underbrace{S(D_j)}_{} \quad , \text{with } x_j \in D_j .$$

note: in 1-dim.: $S(D_j) = \Delta x_j = x_j - x_{j-1}$

With that we can define:

Definition: $f: \bar{D} \to \mathbb{R}$ bounded is **Riemann integrable** on D (or $\bar{D}$) if $\exists\, I \in \mathbb{R}$ s.t.:

$\forall \varepsilon > 0 \; \exists \delta > 0$ s.t. $\forall$ partitions T with $\lambda(T) < \delta$ and $\forall x_j \in D$ we have

$$\left| \sigma(f, T, x_1, \dots, x_k) - I \right| < \varepsilon . \qquad (*)$$

In this case we write $I = \int_D f \, dS$, and $f \in \overbrace{R(D)}^{\text{Riemann integrable on } D}$.

Note: $(*)$ can be expressed as $\lim\limits_{\lambda(T) \to 0} \sigma(f, T) = I$.

Note: We could as well define upper and lower Riemann integrals and call fct.s Riemann integrable if both coincide.

$$\hookrightarrow \underline{\sigma}(f, T) = \sum_j \underbrace{\left(\inf_{x \in D_j} f(x) \right)}_{=: m_j} S(\bar{D}_j) \quad , \quad \overline{\sigma}(f, T) = \sum_j \underbrace{\left(\sup_{x \in D_j} f(x) \right)}_{=: M_j} S(\bar{D}_j)$$

$$\Rightarrow \overline{\int_D} f \, dS := \inf_T \overline{\sigma}(f, T)$$

$$\underline{\int_D} f \, dS := \sup_T \underline{\sigma}(f, T)$$

$\Rightarrow$ If $\overline{\int_D} f \, dS = \underline{\int_D} f \, dS$, then $f \in R(D)$.

$\Rightarrow$ "Riemann criterion": $f \in \mathcal{R}(D) \Leftrightarrow \forall \varepsilon > 0 \; \exists \delta > 0$ s.t. $\sum_j \left| M_j - m_j \right| S(\bar{D}_j) < \varepsilon$

$\forall T$ with $\lambda(T) < \delta$.