Next: Properties of the Riemann integral and connection to repeated 1-dim. integrals.

Recall: $R(D)$ = set of Riemann integrable functions on a domain D.

Theorem:

(i) $R(D)$ is a vector space, $\int$ is linear.

(ii) $R(D)$ is an algebra containing 1, and $\int_D 1\, dS = S(D)$.

i.e., if $f,g \in R(D)$, then $fg \in R(D)$

(iii) $\int$ is monotonic and $m\, S(D) \leq \int_D f\, dS \leq M\, S(D)$, $m = \inf_{x\in D} f(x)$, $M = \sup_{x \in D} f(x)$.

i.e., if $f \leq g$, then $\int_D f\, dS \leq \int_D g\, dS$

(iv) $C(\bar{D}) \subset R(D)$ (Continuous fct.s are Riemann integrable.)

(v) A mean-value theorem (MVT) holds:

If $f \in C(\bar{D})$, then $\exists\, p \in \bar{D}$ s.t. $\int_D f\, dS = f(p)\, S(D)$.

(vi) If $\{D_j\}$ is a partition of D and $f \in R(D)$, then $f \in R(D_j)$ and

$$\int_D f\, dS = \sum_j \int_{D_j} f\, dS.$$

(vii) If $f,g \in R(D)$ and $f = g$ on D, then $\int_D f\, dS = \underbrace{\int_D g\, dS = \int_{\bar{D}} g\, dS}$.

$\int_D g\, dS = \int_{D \cup \partial D} g\, dS$

(viii) If $f \in R(D)$, $g \in C([m,M])$, then $g \circ f \in R(D)$.

(ix) If $f \in R(D)$, then $|f| \in R(D)$ and $\left|\int_D f\, dS\right| \leq \int_D |f|\, dS$.

These properties have nice short proofs, see Kantorovitz. Here, let us just give a

Proof of (ii): Let $f \in R(D)$. We prove that then $f^2 \in R(D)$. This implies:

If $f,g \in R(D) \Rightarrow (f+g)^2, (f-g)^2 \in R(D) \Rightarrow (f+g)^2 - (f-g)^2 = 4fg \in R(D)$
$\Rightarrow fg \in R(D)$.

Left to prove: $f^2 \in R(D)$. Let $\varepsilon > 0$. Then $\exists \delta > 0$ s.t. $\forall T$ with $\lambda(T) < \delta$ we have:

$$\sum_j (M_j - m_j) S(D_j) < \frac{\varepsilon}{2M} \quad \text{(Riemann criterion)}, \text{ with } M = \sup_j M_j .$$

$$\Rightarrow \sum_j \underbrace{(M_j^2 - m_j^2)}_{= (M_j - m_j)\underbrace{(M_j + m_j)}_{\le 2M} \le 2M (M_j - m_j)} S(D_j) \le 2M \sum_j (M_j - m_j) S(D_j) \le 2M \frac{\varepsilon}{2M} = \varepsilon \Rightarrow f^2 \in R(D). \ \square$$

The connection to partial integrals is:

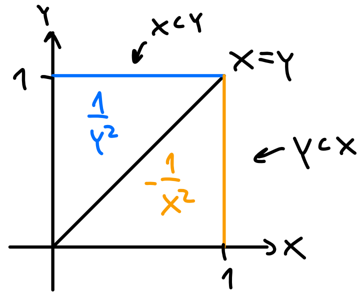
Theorem: For $I = [a,b] \times [\alpha, \beta]$, let $f \in R(I)$ and $f(\cdot, y) \in R([a,b])$ for each $y \in [\alpha, \beta]$.

Then $F(y) := \int_a^b f(x,y)\,dx \in R([\alpha,\beta])$ and $\int_I f\,dS = \int_\alpha^\beta \int_a^b f(x,y)\,dx\,dy$.

Note:

- If $f \notin C(I)$, then order of integration may matter, and even if the iterated integrals exist, f is not necessarily Riemann integrable.
- Just $f \in R(I)$ does not necessarily imply that iterated 1-dim. integrals exist.

Example: Let $f(x,y) := \begin{cases} \frac{1}{y^2} & \text{for } 0<x<y<1 \\ -\frac{1}{x^2} & \text{for } 0<y<x<1 \end{cases}$



For fixed $y>0$: $\int_0^1 f(x,y)\,dx = \int_0^y \frac{1}{y^2}\,dx + \int_y^1 \left(-\frac{1}{x^2}\right)dx = \frac{y}{y^2} + \frac{1}{x}\Big|_y^1 = \frac{1}{y} + 1 - \frac{1}{y} = 1.$

For fixed $x>0$: $\int_0^1 f(x,y)\,dy = \int_0^x \left(-\frac{1}{x^2}\right)dy + \int_x^1 \frac{1}{y^2}\,dy = -\frac{x}{x^2} - \frac{1}{y}\Big|_x^1 = -\frac{1}{x} - 1 + \frac{1}{x} = -1.$

$$\Rightarrow \int_0^1 \int_0^1 f(x,y)\,dx\,dy = 1 \neq -1 = \int_0^1 \int_0^1 f(x,y)\,dy\,dx$$

$\Rightarrow f(\cdot,y), f(x,\cdot) \in \mathcal{R}([0,1])$, so according to thm., $f \notin \mathcal{R}([0,1]^2)$

Note:

- With the previous thm. on exchanging order of integration we get:
  $f \in C(I) \Rightarrow \int_I f\,dS = \int_\alpha^\beta \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_\alpha^\beta f(x,y)\,dy\,dx.$ (In example above: $f \notin C(I)$.)
- Another useful theorem is "Fubini-Tonelli": If $\int_\alpha^\beta \int_a^b |f(x,y)|\,dx\,dy$ exists and is finite, then order of integration can be interchanged. [Check that also this criterion is violated for the example above.]

Next: Connection of Riemann integral to variable boundaries.

Definition: Let $U \subset \mathbb{R}^{n-1}$, let $\Phi, \Psi : U \to \mathbb{R}$ be $C(U)$, define $\mathbb{R}^n \ni x = (x', x_n) = (x_1, \dots, x_{n-1}, x_n)$.
Then $D := \{x \in \mathbb{R}^n : \Phi(x') < x_n < \Psi(x'),\ x' \in U\}$ is called a normal domain.

Note: $S(D) = \int_D 1\,dS = \int_U \int_{\Phi(x')}^{\Psi(x')} dx_n\, \underbrace{dx'}_{\text{or: } d^{n-1}x',\ \text{or } dS(x')} = \int_U (\Psi(x') - \Phi(x'))\,dx'.$

Then:

$\boxed{\underline{\text{Theorem:}}\ \text{If } f \in C(D), \text{ then } \int_D f\, dS = \int_u \int_{\phi(x')}^{\psi(x')} f(x', x_n)\, dx_n\, dx'}$

<u>Sketch of proof:</u> Let $I \subset \mathbb{R}^n$ be a box with $\overline{D} \subset \operatorname{int}(I)$. We extend f by zero to I, i.e., $f|_{I \setminus D} := 0$.

Then $\int_D f\, dS = \int_I f\, dS$, which can be proven by choosing a small enough partition.

(See lemma 4.7.7 in Kantorovitz.)

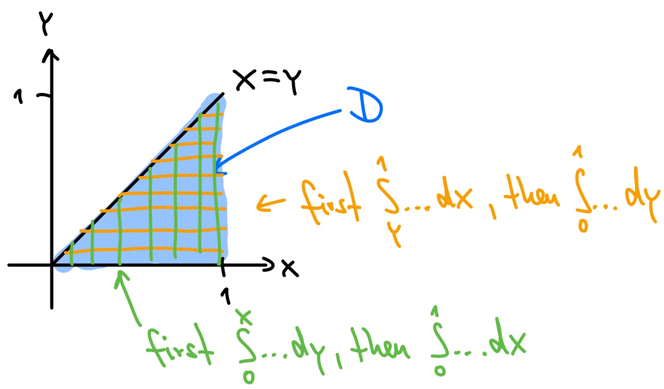
$$\Rightarrow \int_D f\, dS = \int_I f\, dS = \int_{I'} \underbrace{\int_\alpha^\beta f(x', x_n)\, dx_n}_{\substack{\text{1-dim. Riemann integral} \\ = \int_{\phi(x')}^{\psi(x')} f(x', x_n)\, dx_n}}\, dx' = \int_{I'} \int_{\phi(x')}^{\psi(x')} f(x', x_n)\, dx_n\, dx'$$

$$\stackrel{\text{split } \overline{I'} = \overline{u} \cup \overline{I' \setminus u}}{=} \int_u \int_{\phi(x')}^{\psi(x')} f(x', x_n)\, dx_n\, dx' + 0. \quad \square$$

<u>Example:</u> $f(x, y) = y e^{x^3}$.

$$\int_0^1 \int_y^1 y e^{x^3}\, dx\, dy$$

$$\stackrel{\text{by thm. above}}{=} \int_0^1 \underbrace{\int_0^x y e^{x^3}\, dy}_{= e^{x^3} \int_0^x y\, dy = e^{x^3} \frac{y^2}{2}\Big|_{y=0}^{y=x} = \frac{x^2}{2} e^{x^3}}\, dx$$



$$= \int_0^1 \frac{x^2}{2} e^{x^3}\, dx = \int_0^1 \frac{1}{6} \left(\frac{d}{dx} e^{x^3} \right) dx = \frac{1}{6} e^{x^3} \Big|_0^1 = \frac{1}{6}(e - 1).$$

The last important thing we need for the Riemann integral in $\mathbb{R}^n$ is the change of variable formula.

<u>Theorem:</u> Let $U, V \subset \mathbb{R}^n$ be domains with content, let $\phi: U \to V$ be a diffeomorphism (i.e., $\phi \in C^1$, ϕ invertible, and $\phi^{-1} \in C^1$). Then, for $f \in R(V)$ we have

$$\boxed{\underbrace{\int_V f\, dx}_{= \int_V f\, dS} = \underbrace{\int_U f(\phi(u))\, |\det D\phi(u)|\, du}_{= \int_U f \circ \phi\, |\det D\phi|\, dS}.}$$

(We skip the proof.)

Note:

- Think of $|\det D\phi(u)|\, du$ as the transformed volume element, keeping in mind that the determinant is a volume!
- In 1-dim. we know the substitution formula

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\phi(u))\, \phi'(u)\, du \quad , \text{for } a<b \ , \ \phi: [\alpha, \beta] \to [a,b] \, , \ \phi' > 0.$$

Note that when $\phi' < 0$, then $\int_a^b f(x)dx = \int_\beta^\alpha f(\phi(u))\, \phi'(u)\, du = \int_\alpha^\beta f(\phi(u))(-\phi'(u))\, du$,

so indeed $\int_a^b f(x)dx = \int_\alpha^\beta f(\phi(u))\, |\phi'(u)|\, du$, as in the theorem.

<u>Important examples:</u>

Polar coordinates ($\mathbb{R}^2$): $\phi(r, \varphi) = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix}$

$\Rightarrow D\phi(r,\varphi) = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix}$, so $\det(D\phi(r,\varphi)) = r\cos^2\varphi + r\sin^2\varphi = r$.

$$\Rightarrow \int_{B_R(0)} f(x)\,dx = \int_0^R \int_0^{2\pi} f(r\cos\varphi, r\sin\varphi)\,d\varphi\, r\,dr.$$

- E.g., area of a circle: $\int_{B_R(0)} 1\,dx = \int_0^R \int_0^{2\pi} 1\,d\varphi\, r\,dr = 2\pi \int_0^R r\,dr = \pi R^2.$

- E.g., area of an ellipse $E := \{(x,y)\in\mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1\}$, given $a,b>0$.

  We can def. $\Phi: B_1(0) \to E$, $\Phi(u,v) = \begin{pmatrix} au \\ bv \end{pmatrix} \Rightarrow D\Phi = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \Rightarrow \det D\Phi = ab$

  $$\Rightarrow \int_E dx = \int_{B_1(0)} ab\, d(u,v) = ab\pi.$$

- E.g., Gaussian integral: $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$

  Trick: $I^2 = \int_{-\infty}^{\infty} e^{-x^2}dx \int_{-\infty}^{\infty} e^{-y^2}dy = \int_{\mathbb{R}^2} e^{-x^2-y^2}\,d(x,y) := \lim_{R\to\infty} \int_{B_R(0)} e^{-x^2-y^2}\,d(x,y)$

  $$= \int_0^\infty \int_0^{2\pi} e^{-r^2}\,d\varphi\, r\,dr$$

  $$= 2\pi \int_0^\infty \underbrace{e^{-r^2} r}_{= \frac{-1}{2}\left(\frac{d}{dr} e^{-r^2}\right)}\,dr$$

  $$= -\pi\, e^{-r^2}\Big|_0^\infty$$

  $$= \pi$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$