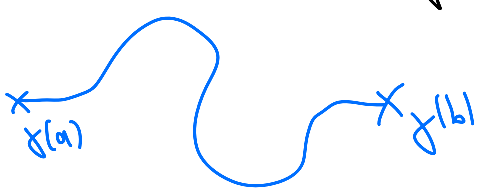
3.3 Line Integrals

Next: We consider integrals along curves and surfaces and their relation to Riemann integrals and each other. This will lead us to generalizations of the Fundamental Theorem of Calculus. (E.g., $\int_{\text{curve } \gamma} F \, dx$ depends only on endpoints $\gamma(a)$ and $\gamma(b)$. E.g., $\underbrace{\int_D \nabla \cdot G \, dS = \int_{\partial D} G \cdot n \, ds}_{\text{integral over } D \text{ depends only on } \partial D!}$.)

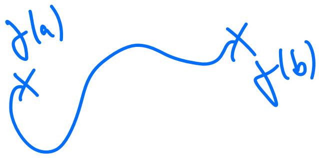
Applications: Force fields, electrodynamics, ...

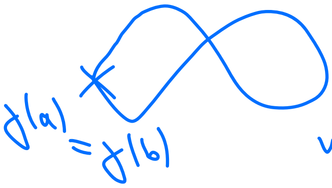
Definition:
Any continuous function $\gamma : [a,b] \to \mathbb{R}^n$ is called an $\boxed{\text{oriented curve}}$ (or path).
i.e., $\gamma \in C([a,b], \mathbb{R}^n)$



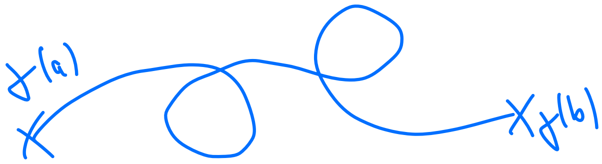
First, a few important types of curves:

- If $\gamma(a) = \gamma(b)$, the curve is $\boxed{\text{closed}}$.
- If $\gamma : [a,b] \to \mathbb{R}^n$ is injective, the curve is $\boxed{\text{simple}}$.





not simple, closed



not simple, not closed

- Two curves γ and ϱ are called **equivalent** if there is a continuous, monotonic, increasing h s.t. $\underbrace{\gamma = \varrho \circ h}_{\text{"going through the curve with a different velocity"}}$ (i.e., the images of γ and $\varrho \circ h$ are the same).

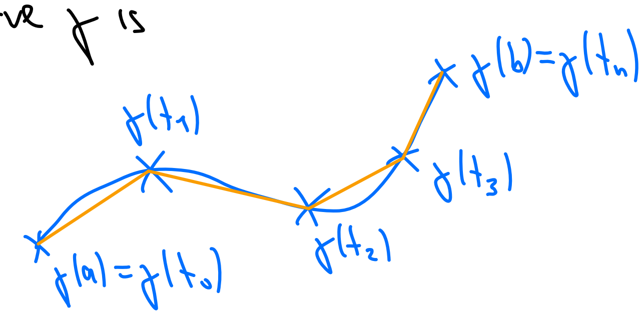
Next, we define the length of a curve (not all curves have a length!)

Let $\mathcal{T}$ be a partition of $[a,b]$ with $a = t_0 < t_1 < t_2 < \dots < t_n := b$, and let

$$\lambda(\mathcal{T}) := \max_{i=1,\dots,n} \underbrace{|t_i - t_{i-1}|}_{=:\Delta t_i}$$

Then an approximation to the length of a curve γ is

$$\Lambda(\mathcal{T},\gamma) := \sum_{i=1}^{n} \|\gamma(t_i) - \gamma(t_{i-1})\|.$$



length of γ ≈ sum of lengths of line segments

<u>Definition:</u> The **length of the curve** $\gamma \in C([a,b])$ is defined as

$$\Lambda(\gamma) = \sup_{\mathcal{T}} \Lambda(\mathcal{T},\gamma).$$

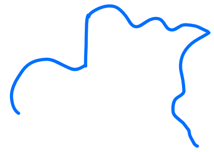
If $\Lambda(\gamma) < \infty$, we call γ **rectifiable** ("γ has length").

We get a more concrete formula for continuously differentiable curves.

Theorem: Let $\gamma \in C^1([a,b])$. Then γ is rectifiable and

$$\boxed{\Lambda(\gamma) = \int_a^b \|\gamma'(t)\|\, dt.}$$

Note: The theorem is obviously extended to piece-wise C^1 curves.



Proof:

"$\leq$": $\Lambda(\mathcal{J},\gamma) = \sum_{i=1}^{n} \|\gamma(t_i) - \gamma(t_{i-1})\| = \sum_{i=1}^{n} \left\| \int_{t_{i-1}}^{t_i} \gamma'(t)\, dt \right\|$

$$\leq \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \|\gamma'(t)\|\, dt = \int_a^b \|\gamma'(t)\|\, dt,$$

so also $\Lambda(\gamma) = \sup_{\mathcal{J}} \Lambda(\mathcal{J},\gamma) \leq \int_a^b \|\gamma'(t)\|\, dt.$

"$\geq$": Let $\varepsilon > 0$. We know that γ' is uniformly continuous $\Rightarrow \exists \delta > 0$ s.t. $\forall s,t \in [a,b]$ with $|s-t| < \delta$ we have $\|\gamma'(s) - \gamma'(t)\| < \varepsilon$.

Let $\mathcal{J}$ be a partition with $\lambda(\mathcal{J}) < \delta$.

Then $\|\gamma'(t)\| \leq \|\gamma'(t_i)\| + \varepsilon \quad \forall t \in [t_{i-1}, t_i]$

$$\Rightarrow \int_a^b \|\gamma'(t)\|\, dt = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \|\gamma'(t)\|\, dt \leq \sum_{i=1}^{n} \left(\|\gamma'(t_i)\| + \varepsilon \right) \Delta t_i$$

$$= \sum_{i=1}^{n} \left(\Big\| \underbrace{\int_{t_{i-1}}^{t_i} \gamma'(t_i)\, dt}_{} \Big\| + \varepsilon \Delta t_i \right)$$

$$= \int_{t_{i-1}}^{t_i} \gamma'(t)\, dt + \int_{t_{i-1}}^{t_i} \underbrace{(\gamma'(t_i) - \gamma'(t))}_{\leq \varepsilon \text{ in abs. value}}\, dt$$

$$\le \sum_{i=1}^{n} \Big(\underbrace{\Big\| \int_{t_{i-1}}^{t_i} \gamma'(t)\,dt \Big\|}_{= \gamma(t_i) - \gamma(t_{i-1})} + 2\varepsilon \Delta t_i \Big)$$

$$= \underbrace{\Lambda(\sigma, \gamma)}_{\le \Lambda(\gamma)} + 2\varepsilon (b-a)$$

Since ε was arbitrary (arbitrarily small), we find $\int_a^b \|\gamma'(t)\| dt \le \Lambda(\gamma)$. $\square$

Note that for $\gamma \in C^1$, the length $\Lambda(\gamma)$ is independent of the parameterization: If $\gamma = \varrho \circ h$, $h \in C^1$ increasing, then

$$\int_a^b \|\gamma'(t)\| dt = \int_a^b \Big\| \frac{d}{dt} \varrho(h(t)) \Big\| dt \underset{\text{chain rule}}{=} \int_a^b \|\varrho'(h(t))\| \, h'(t)\,dt \underset{\text{substitution}}{=} \int_{h(a)}^{h(b)} \|\varrho'(u)\| du.$$

$(u = h(t) \Rightarrow du = h'(t)\,dt)$

Examples:

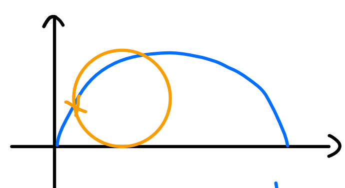
- Non-rectifiable curve: see homework.
- Circumference of a circle: $\gamma(t) = R(\cos t, \sin t)$, $t \in [0, 2\pi]$

  $\Rightarrow \gamma'(t) = R(-\sin t, \cos t) \Rightarrow \|\gamma'(t)\| = \sqrt{R^2 \sin^2 t + R^2 \cos^2 t} = R.$

  $\Rightarrow \Lambda(\gamma) = \int_0^{2\pi} R\,dt = 2\pi R.$

- Cycloid: $\gamma(t) = (t - \sin t, 1 - \cos t)$, $t \in [0, 2\pi]$

  $\Rightarrow \gamma'(t) = (1 - \cos t, \sin t)$



"move on a circle that is moved along x-axis"

$$\Rightarrow \|\gamma'(t)\|^2 = (1-\cos t)^2 + \sin^2 t = 1 - 2\cos t + \cos^2 t + \sin^2 t = 2(1-\cos t)$$

standard trigonometric identity $\;= 4\sin^2\left(\frac{t}{2}\right)$

$$\Rightarrow \Lambda(\gamma) = \int_0^{2\pi} 2\sin\left(\frac{t}{2}\right) dt = -4\cos\left(\frac{t}{2}\right)\Big|_0^{2\pi} = 4-(-4) = 8.$$

One useful parametrization is the arclength parametrization:

Define $s(t) = \int_a^t \|\gamma'(\tau)\| \, d\tau$. Then $s'(t) = \|\gamma'(t)\| > 0$ for $\gamma'(t) \neq 0$

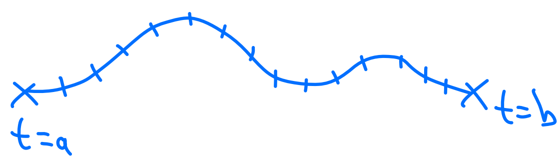
("non-degenerate parametrization").

Since $s(t)$ is monotonic it is invertible, with inverse $t(s)$.

We call $\gamma(t(s))$ the **arclength parametrization**.

$$\Rightarrow \frac{d}{ds}\gamma(t(s)) = \gamma'(t(s))\frac{dt(s)}{ds} = \gamma'(t(s))\frac{1}{s'(t(s))} \Rightarrow \left\|\frac{d}{ds}\gamma(t(s))\right\| = \left\|\frac{\gamma'}{\|\gamma'\|}\right\| = 1,$$

i.e., we go through the curve with speed 1.



Definition: For $f \in C(\gamma, \mathbb{R})$, and γ a C^1 curve, we define the

line integral $\displaystyle\int_\gamma f \, ds = \int_0^{\Lambda(\gamma)} f(\gamma(t(s))) \, ds.$

Note: $\displaystyle\int_\gamma f ds := \int_0^{\Lambda(\gamma)} f(\gamma(t(s))) \, ds = \int_a^b f(\gamma(t)) \, \|\gamma'(t)\| \, dt.$

Substitution $t = t(s) \Rightarrow \frac{dt}{ds} = \frac{1}{s'(t)} = \frac{1}{\|\gamma'(t)\|} \Rightarrow ds = \|\gamma'(t)\| dt$

Note also that $\int_\gamma f \, ds$ is independent of the parametrization of γ (see HW).

Next: How to def. line integrals for vector fields $F: D \subset \mathbb{R}^n \to \mathbb{R}^n$?

(E.g., force fields in physics.)

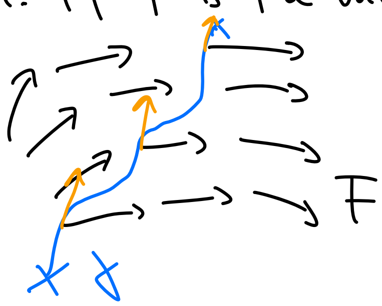
In physics: work = force $\cdot$ $\underbrace{\text{length}}_{\text{or rather "displacement"}}$.

In 1-dim.: $W = \int F(s)\,ds$.

In n-dimensions: Only displacement in the direction of the force is work (e.g., displacement orthogonal to force causes no work).

In general: If T is the unit tangent vector, then work $= \langle F, T \rangle$.

$T = \frac{\gamma'}{\|\gamma'\|}$



$$\Rightarrow \text{Total work } W = \int_\gamma \langle F, T \rangle\, ds = \int_\gamma \langle F, \frac{\gamma'}{\|\gamma'\|} \rangle\, ds$$

$$= \int_a^b \langle F(\gamma(t)), \frac{\gamma'(t)}{\|\gamma'(t)\|} \rangle\, \|\gamma'(t)\|\, dt$$

$$= \int_a^b \langle F \circ \gamma, \gamma' \rangle\, dt$$

$$= \int_a^b F(\gamma(t)) \cdot \gamma'(t)\, dt.$$

$$\boxed{\begin{array}{l}\underline{\text{Definition:}} \text{ For } F \in C(\gamma, \mathbb{R}^n) \text{ and } \gamma \text{ a } C^1 \text{ curve, we define} \\ \text{the line integral } \int_\gamma F\, ds = \int_a^b F(\gamma(t))\, \gamma'(t)\, dt.\end{array}}$$

Note: $F\,dx := F_1 dx_1 + \ldots + F_n dx_n$ is called a first-order differential form.