Recall from last time:

- The length of a curve $\gamma \in C^1([a,b],\mathbb{R}^n)$ is $\Lambda(\gamma) = \int_a^b \|\gamma'(t)\| dt$.
- For $f \in C(\gamma,\mathbb{R})$, $\gamma \in C^1([a,b],\mathbb{R}^n)$, we def. the line integral

$$\int_\gamma f\,dx := \int_a^b f(\gamma(t))\|\gamma'(t)\|dt.$$

(Note: For $f=1$ this yields $\int_\gamma dx = \Lambda(\gamma)$.)

- For $F \in C(\gamma,\mathbb{R}^n)$ (a vector field), $\gamma \in C^1([a,b],\mathbb{R}^n)$, we def. the line integral

$$\int_\gamma F\,dx := \int_a^b \underbrace{F(\gamma(t))\cdot\gamma'(t)}_{\text{scalar product of two vectors}}dt.$$

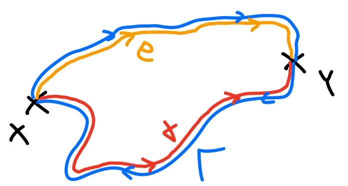
Generally, the value of $\int_\gamma F\,dx$ might depend on all of γ. But sometimes it might just depend on $\gamma(a)$ and $\gamma(b)$ (and not on $\gamma(t)$ for $a<t<b$).

Definition: Let $F \in C(D,\mathbb{R}^n)$, $D \subset \mathbb{R}^n$ a domain, γ piecewise $C^1([a,b],\mathbb{R}^n)$ ("piecewise smooth"). Then we call F **conservative** if $\int_\gamma F\,dx$ depends only on $\gamma(a)$ and $\gamma(b)$.

Note: In the differential form language: F conservative $\Longleftrightarrow$ $F\,dx$ **exact**.

Lemma: F conservative $\Longleftrightarrow$ $\int_\gamma F\,dx = 0$ $\forall$ closed γ.

Proof:



$$\int_\Gamma F\,dx = \int_e F\,dx - \int_\gamma F\,dx$$

If $\int_\gamma F\,dx$ depends only on $\gamma(a)=x$ and $\gamma(b)=y$, then $\int_\rho F\,dx = \int_\gamma F\,dx$, since $\gamma(a)=\rho(a)$, $\gamma(b)=\rho(b)$. Thus $\int_\Gamma F\,dx = 0$.

If $\int_\Gamma F\,dx = 0$, then $\int_\rho F\,dx = \int_\gamma F\,dx \ \forall \rho$, so $\int_\gamma F\,dx$ depends only on $\gamma(a), \gamma(b)$. $\square$

From physics: work $\int_\gamma F\,dx$ should only depend on $\gamma(a), \gamma(b)$ if F comes from a potential, i.e., $F = \nabla\phi$, $\phi: \mathbb{R}^n \to \mathbb{R}$. Indeed:

Theorem: $F \in C(D, \mathbb{R}^n)$ ($D \subset \mathbb{R}^n$ a domain) is conservative if and only if $\exists\, \phi \in C^1(D, \mathbb{R})$ s.t. $F = \nabla\phi$. (ϕ is called a "potential" for F.)

Proof:

"$\Leftarrow$" This direction is a direct computation:

$$\int_\gamma F\,dx := \int_a^b F(\gamma(t)) \cdot \gamma'(t)\,dt = \int_a^b (\nabla\phi)(\gamma(t))\,\gamma'(t)\,dt \underset{\text{chain rule}}{=} \int_a^b \frac{d}{dt}\big(\phi(\gamma(t))\big)\,dt \underset{\text{Fundamental Theorem of Calculus}}{=} \phi(\gamma(a)) - \phi(\gamma(b)).$$

$\Rightarrow F$ conservative.

(Note: For γ piecewise smooth we split it into a sum of C^1 curves first.)

"$\Rightarrow$" We construct ϕ directly. If F conservative, we fix some $x_0 \in D$ and define

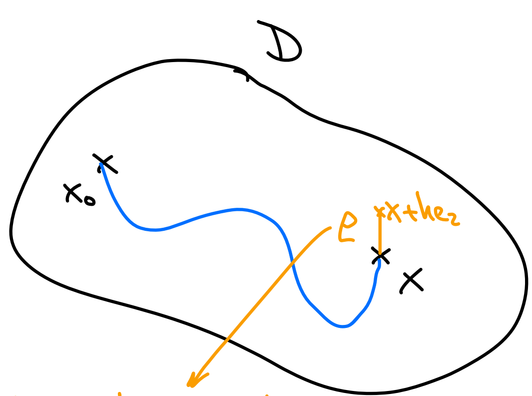
$\phi(x) = \int_\gamma F\,dx$, where γ is any C^1 curve with $\gamma(a) = x_0$, $\gamma(b) = x$. (If F were not conservative, ϕ would not just be a fct. of x.)

Then we check:

$$\phi(x + he_i) = \int_{\gamma \cup \rho} F\,dx = \int_\gamma F\,dx + \int_\rho F\,dx$$

$$= \phi(x) + \int_0^h F(x + te_i) \cdot e_i\,dt$$

$$= \phi(x) + \int_0^h F_i(x + te_i)\,dt$$



$\rho(t) = x + te_i$, $0 \le t \le h$
$\Rightarrow \rho'(t) = e_i$

$$\Rightarrow \frac{\partial \Phi}{\partial x_i} := \lim_{h\to 0} \frac{\Phi(x+he_i)-\Phi(x)}{h} = \frac{d}{dh}\Phi(x+he_i)\Big|_{h=0} = F_i(x+he_i)\Big|_{h=0} = F_i(x),$$

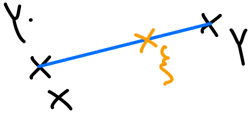
i.e., $\nabla\Phi = F$.

Since F continuous, $\Phi \in C^1$. $\square$

Note:

- If $\Phi(x)-\Psi(x) = \text{const}$ on D, then $\nabla\Phi = \nabla\Psi$ on D.
- Let $F = \nabla\Phi = \nabla\Psi$. For any $\Theta \in C^1(D)$, the mean value thm. tells us that

  $\Theta(x)-\Theta(y) = \nabla\Theta(\xi)(x-y)$ for some ξ on straight line between x and y.

  Since D is a domain, it is connected, i.e., any two points can be connected by a polygonal path.

  So for $\Theta = \Phi - \Psi$ we have $\nabla\Theta = 0$ and thus $\Theta = \Phi-\Psi = \text{const}$ along any straight line segment, i.e., $\Phi - \Psi = \text{const}$ on D.

$\Rightarrow$ $F = \nabla\Phi = \nabla\Psi$ on D $\Leftrightarrow$ $\Phi - \Psi = \text{const}$ on D.

An immediate consequence of the thm. above is:

Corollary: If $F \in C^1(D,\mathbb{R}^n)$ ($D \subset \mathbb{R}^n$ a domain) is conservative, then the derivative DF is symmetric.

Proof: F conservative $\Rightarrow F = \nabla\Phi \Rightarrow DF = D(\nabla\Phi) = H_\Phi$ i.e., $(DF)_{ij} = \frac{\partial \Phi}{\partial x_i \partial x_j}$, which is symmetric (for $F \in C^1$, i.e., $\Phi \in C^2$). $\square$

Question for next time: Is "DF symmetric" also sufficient for F to be conservative?

Not always...