Advanced Calculus and Methods of Mathematical Physics — Session 21

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Recall from last time:

- Let $F \in C(D, \mathbb{R}^n)$ ($D \subset \mathbb{R}^n$ a domain). Then:

F conservative $\iff \int_\gamma F\,dx$ depends only on $\gamma(a), \gamma(b)$ for any C^1 curve $\gamma : [a,b] \to \mathbb{R}^n$

$\iff \int_\gamma F\,dx \;\; \forall$ closed curves γ

$\iff \exists\, \phi \in C^1(D, \mathbb{R}^n)$ s.t. $F = \nabla \phi$

- $F \in C^1$ conservative $\Rightarrow \underbrace{DF}_{\text{the Jacobian of } F}$ ($= H_\phi$, the Hessian of ϕ from $F = \nabla\phi$) is symmetric

Example to show that "DF symmetric" is not sufficient for F to be conservative:

$$F = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right) \text{ on } D = \mathbb{R}^2 \setminus \{0\}.$$

Here: $$\frac{\partial F_1}{\partial y} = \frac{-1}{x^2+y^2} - y\frac{(-2y)}{(x^2+y^2)^2} = \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2},$$

and $$\frac{\partial F_2}{\partial x} = \frac{1}{x^2+y^2} + x\frac{(-2x)}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2},$$

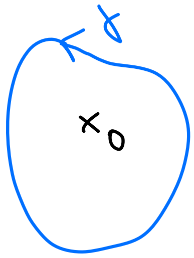
so $DF = \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix}$ is symmetric.

But: let $\gamma : [0, 2\pi] \to \mathbb{R}^2$, $t \mapsto (\cos t, \sin t)$ (unit circle).

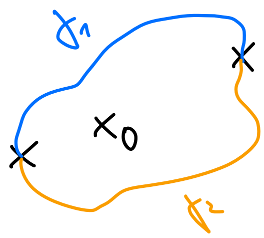
Then $$\int_\gamma F \cdot dx = \int_0^{2\pi} \underbrace{(-\sin t, \cos t)}_{F(\gamma(t))} \underbrace{\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}}_{\gamma'(t)} dt = \int_0^{2\pi} \underbrace{(\sin^2 t + \cos^2 t)}_{=1}\, dt = 2\pi.$$

$\Rightarrow F$ not conservative!

The problem here is the topological shape of the domain $D = \mathbb{R}^2 \setminus \{0\}$. The "hole" at 0 makes it not "simply connected", where "simply connected" means: any closed curve can be continuously contracted to a point (or equivalently: any two paths with same start/end points can be continuously deformed into each other, keeping the start/end points fixed).



If 0 is missing, γ cannot be cont. deformed to a point.

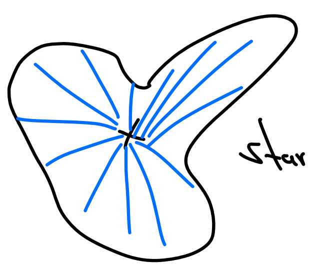


γ_1 cannot be cont. deformed into γ_2 (keeping start/end points fixed) if 0 is missing

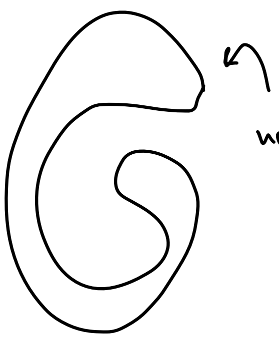
Generally, if D is simply connected, then $F : D \to \mathbb{R}^n$, $F \in C^1$ with DF symmetric implies that F is conservative.

Here, we prove this for a special case.

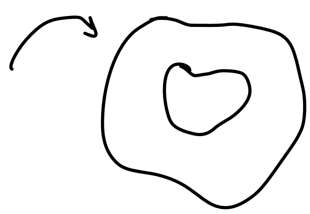
<u>Definition</u>: $D \subset \mathbb{R}^n$ is called star-shaped if $\exists\, p \in D$ (the star center) such that any $x \in D$ can be connected to p by a straight line segment.



star-shaped



not star-shaped



(Any non-empty convex set is star-shaped.)

<u>Theorem</u>: Let $D \subset \mathbb{R}^n$ be star-shaped. Then:

$$F \in C^1(D, \mathbb{R}^n) \text{ conservative} \iff DF \text{ symmetric}$$

Proof: We need to show "$\Leftarrow$". Let 0 be the center of D (without loss of generality).

Let γ be the straight line segment from 0 to x, i.e., $\gamma : [0,1] \to \mathbb{R}^n, t \mapsto tx$. Then we def.

$$\phi(x) := \int_\gamma F \cdot dx = \int_0^1 \underbrace{F(tx)}_{F(\gamma(t))} \cdot \underbrace{x}_{\gamma'(t)} \, dt.$$

$$\Rightarrow \frac{\partial \phi}{\partial x_i} \underset{\text{product rule}}{=} \int_0^1 \Big(\frac{\partial F}{\partial x_i}(tx) t \cdot x + F(tx) \cdot e_i \Big) dt = \int_0^1 \frac{d}{dt}\big(tF_i(tx)\big) dt = F_i(x) - 0.$$

$$\frac{\partial F}{\partial x_i}(tx) t \cdot x + F(tx) \cdot e_i = \sum_j \underbrace{\frac{\partial F_j}{\partial x_i}}_{= \frac{\partial F_i}{\partial x_j}(tx)}(tx) x_j t + F(tx) \cdot e_i = \frac{d}{dt}\big(tF_i(tx)\big)$$

$= \frac{\partial F_i}{\partial x_j}(tx)$ by assumption that DF is symmetric

$\Rightarrow D\phi = F$, i.e., F is conservative. $\square$

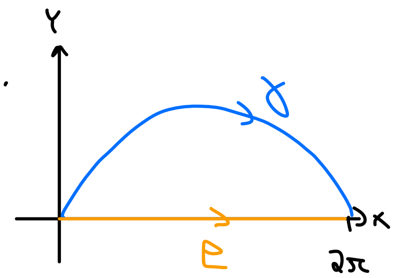
Examples:

- Let $F(x,y) = \left(\frac{y^2}{1+x^2}, 2y \arctan x \right)$. Task: Compute $\int_\gamma F \, dx$, e.g., for γ an ellipse.

  Here, $\frac{\partial F_1}{\partial y} = \frac{2y}{1+x^2}$, and $\frac{\partial F_2}{\partial x} = 2y \frac{1}{1+x^2}$ so DF is symmetric on $\mathbb{R}^2$.

  $\Rightarrow \int_\gamma F \, dx = 0$ for any closed γ.

- Let $\gamma : [0, 2\pi] \to \mathbb{R}^2, t \mapsto (t - \sin t, 1 - \cos t)$ be the cycloid.



  Let $e : [0, 2\pi] \to \mathbb{R}^2, t \mapsto (t, 0)$.

  $F = \frac{2}{1+x^2+y^2}(x,y)$ is conservative on $\mathbb{R}^2$, as can be checked.

$$\Rightarrow \underbrace{\int_\gamma F \cdot dx}_{\text{complicated to compute}} = \underbrace{\int_e F \cdot dx}_{\text{easier to compute}} = \int_0^{2\pi} F(e(t)) e'(t) dt = \int_0^{2\pi} \frac{2}{1+t^2}(t,0) \cdot \binom{1}{0} dt = \int_0^{2\pi} \frac{2t}{1+t^2} dt$$

$$= \ln(1+t^2) \Big|_0^{2\pi} = \ln(1+4\pi^2) - \ln(1) = \ln(1+4\pi^2).$$