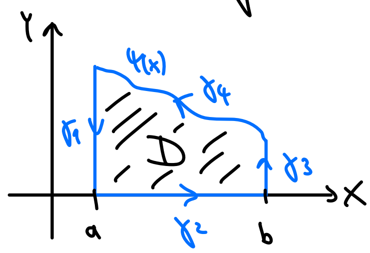
3.4 Green's Theorem

Green's thm. relates integrals over bounded closed domains $\overline{D} \subset \mathbb{R}^2$ to line integrals over the boundary ∂D.

An example as motivation (and partly sketch of proof):

Consider

- an x-normal domain $D = \{(x,y) : x \in (a,b),\ 0 \le y \le \varphi(x)\}$,
- a C^1 vector field $F = (f,g)$,
- a curve γ:



i.e., $\gamma = \partial D$ (going anti-clockwise).

Then $\int_D \left(-\frac{\partial f}{\partial y}\right) dS = \int_a^b \underbrace{\int_0^{\varphi(x)} \left(-\frac{\partial f}{\partial y}\right) dy}_{= f(x,0) - f(x,\varphi(x))} dx = \int_a^b f(x,0)\,dx - \int_a^b f(x,\varphi(x))$

Also: $\int_\gamma f\,dx = \int_\gamma (f,0)\cdot \underbrace{dx}_{\substack{d(x,y) \text{ might be} \\ \text{a better notation}}} = \underbrace{\int_{\gamma_1} (f,0)\cdot dx}_{=0} + \int_{\gamma_2} (f,0)\cdot dx + \underbrace{\int_{\gamma_3} (f,0)\cdot dx}_{=0} + \int_{\gamma_4} (f,0)\cdot dx.$

(since $(f,0)$ perpendicular to γ_1)

Now:
- $\int_{\gamma_2} (f,0)\cdot d(x,y) = \int_a^b (f(x,0),0)\cdot(1,0)\,dx = \int_a^b f(x,0)\,dx$
- $\int_{\gamma_4} (f,0)\cdot d(x,y) = \int_b^a (f(x,\varphi(x)),0)\cdot(1,\varphi'(x)) = -\int_a^b f(x,\varphi(x))\,dx$

$$\Rightarrow \int_{D} \left(-\frac{\partial f}{\partial y}\right) dS = \int_{\gamma} f \, dx$$

no minus sign

A similar computation holds for g, namely $\int_{D} \left(\frac{\partial g}{\partial x}\right) dS = \int_{\gamma} g \, dy$ [check this]

Together, we find $\int_{D} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dS = \int_{\gamma} F \cdot d(x,y)$.

Note: Interchanging x and y gives same result for y-normal domains.

More generally, let us define:

Definition: $D \subset \mathbb{R}^2$ bounded is called a **regular domain** if it can be decomposed into finitely many $\underbrace{\text{bi-normal}}_{\text{either } x\text{- or } y\text{-normal}}$ subdomains.

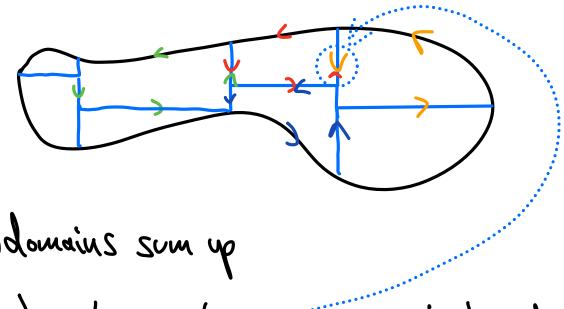
Then the result from above still holds:

Theorem (Green's theorem):
Let $D \subset \mathbb{R}^2$ be a bounded regular domain, and let $F \in C^1(\bar{D}, \mathbb{R}^2)$. Then

$$\boxed{\int_{D} \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right) dS = \int_{\partial D} F \cdot dx \qquad \text{(Green's formula)},}$$

where the line integral has anti-clockwise orientation.

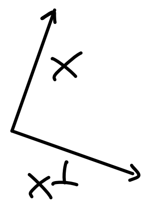
Sketch of proof:



- area integrals over subdomains sum up
- interior pieces of line integrals cancel $\longrightarrow$ only boundary pieces remain
- Summing up yields the thm. $\square$

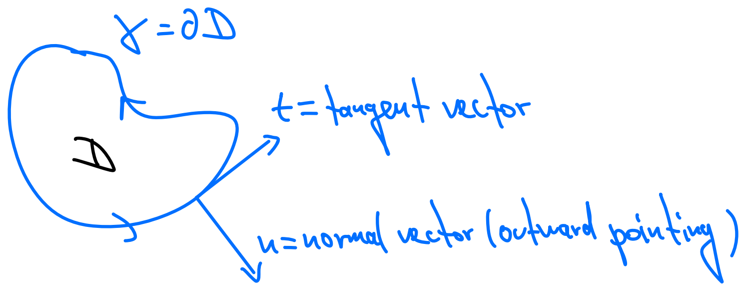
Remarks:

- For $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, def. $x^\perp = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$.



With $\nabla^\perp := \begin{pmatrix} -\frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} \end{pmatrix}$, Green's formula becomes $\int_D \underbrace{\nabla^\perp \cdot F}_{=\mathrm{curl}_{2D} F} \, dS = \int_{\partial D} F \cdot dx$.

- Def. vectors n and t as in the picture:



Let us define G s.t. $F = G^\perp$.

Then $\int_D \nabla \cdot G \, dS = \int_D \nabla^\perp G^\perp \, dS = \int_D \nabla^\perp \cdot F = \int_{\partial D} F \cdot dx = \int_{\partial D} \underbrace{G^\perp t}_{=\underbrace{(G^\perp)^\perp}_{=-G} \cdot \underbrace{t^\perp}_{=n} = -G \cdot n} ds$

$\Rightarrow \boxed{\int_D \underbrace{\nabla \cdot G}_{=\mathrm{div}\, G \text{ (divergence of } G)} \, dS = \int_{\partial D} G \cdot n \, ds}$ (Divergence Theorem)

(Generalization of the Fundamental Thm. of Calculus to 2-dim. domains)

Examples:

- $F = \frac{1}{2}\begin{pmatrix}-y\\ x\end{pmatrix} = \frac{1}{2}x^{\perp} \Rightarrow \nabla^{\perp}\cdot F = \frac{1}{2}\big(1-(-1)\big) = 1$

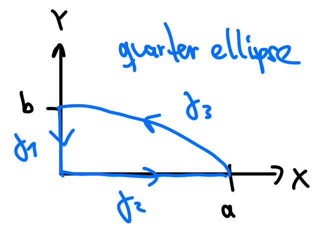
$$\Rightarrow \int_D \nabla^{\perp}\cdot F\, dS = \int_D dS = \underbrace{S(D)}_{\text{volume of } D} \overset{\text{Green's thm.}}{=} \int_{\partial D} F\cdot dx = \frac{1}{2}\int_{\partial D} x\,dy - \frac{1}{2}\int_{\partial D} y\,dx$$

E.g., area of ellipse D, with ∂D parametrized by $\gamma(t) = (a\cos t, b\sin t)$, $t\in[0,2\pi]$:

$$S(D) = \int_{\partial D} F\cdot dx = \int_0^{2\pi} F(\gamma(t))\cdot\gamma'(t)\,dt = \frac{1}{2}\int_0^{2\pi} \gamma^{\perp}(t)\cdot\gamma'(t)\,dt = \frac{1}{2}ab\,2\pi = \pi ab$$

$$= (-b\sin t, a\cos t)\cdot(-a\sin t, b\cos t) = ab\sin^2 t + ab\cos^2 t = ab$$

- $J = \int_D xy\,dS$ with $D = \left\{(x,y)\in\mathbb{R}^2 : \frac{x^2}{a^2}+\frac{y^2}{b^2} = 1,\ x\geq 0, y\geq 0\right\}$



To use Green's thm. we can choose, e.g., $F = (0, \frac{1}{2}x^2y)$, s.t. $\nabla^{\perp}\cdot F = xy$.

$$\text{Then } J = \int_D \nabla^{\perp}\cdot F\,dS = \int_{\partial D} F\cdot dx = \underbrace{\int_{\gamma_1} F\cdot dx}_{\substack{=0 \\ (\text{since } x=0)}} + \underbrace{\int_{\gamma_2} F\cdot dx}_{\substack{=0 \\ (\text{since } y=0)}} + \int_{\gamma_3} F\cdot dx$$

$$= \int_0^{\frac{\pi}{2}} \left(0, \tfrac{1}{2}(a\cos t)^2\, b\sin t\right)\cdot(-a\sin t, b\cos t)\,dt$$

$$= \int_0^{\frac{\pi}{2}} \tfrac{1}{2}a^2b^2\cos^3 t\,\sin t\,dt$$

$$= \tfrac{1}{2}a^2b^2\left(-\tfrac{1}{4}\right)\cos^4 t\,\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{a^2b^2}{8}$$

[Note: direct computation (without using Green's thm.) is also possible here; check this]