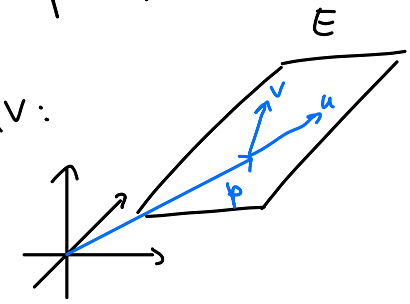
3.5 Surface Integrals

First, a short review of planes, normal vectors, and the cross product.



A plane E can be parametrized by specifying vectors p, u, v:

$$E = \{ x \in \mathbb{R}^3 : x = p + su + tv, \ s,t \in \mathbb{R} \}$$

This can be written as

$$\underbrace{\begin{pmatrix} | & | & | \\ p-x & u & v \\ | & | & | \end{pmatrix}}_{=A} \begin{pmatrix} 1 \\ s \\ t \end{pmatrix} = 0$$

(column vectors)

with A $\underbrace{\text{singular}}_{\Leftrightarrow\ Ax = 0 \text{ has a solution } x \neq 0}$, i.e., $\det A = 0$.

Now recall the Laplace expansion from linear Algebra:

$$\det \begin{pmatrix} (p-x)_1 & u_1 & v_1 \\ (p-x)_2 & u_2 & v_2 \\ (p-x)_3 & u_3 & v_3 \end{pmatrix} = (p-x)_1 \underbrace{\det \begin{pmatrix} u_2 & v_2 \\ u_3 & v_3 \end{pmatrix}}_{= u_2 v_3 - u_3 v_2 =: n_1} + (p-x)_2 \underbrace{(-1) \det \begin{pmatrix} u_1 & v_1 \\ u_3 & v_3 \end{pmatrix}}_{= u_3 v_1 - u_1 v_3 =: n_2} + (p-x)_3 \underbrace{\det \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}}_{= u_1 v_2 - u_2 v_1 =: n_3}$$

$$=: (p-x) \cdot n$$

Thus, $\boxed{n = u \times v \ \text{(cross product)}}$, and the eq. of a plane can be written as

$$(p-x) \cdot (u \times v) = 0.$$

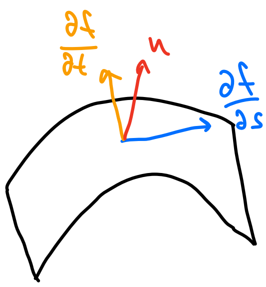
Note/recall the properties of the cross product:

- $u \times v = - v \times u$
- $\det \begin{pmatrix} | & | & | \\ a & b & c \\ | & | & | \end{pmatrix} = a \cdot (b \times c)$
- $u \times v$ is perpendicular to u and v

- $u \times v = 0$ if u and v are linearly dependent
- The area of a parallelogram spanned by u and v is $\|u \times v\| = \|u\|\,\|v\| \sin\theta$, with θ = angle between u, v
- $\|u \times v\|^2 = \|u\|^2 \|v\|^2 - (u \cdot v)^2$

Next: More generally, a surface $M \subset \mathbb{R}^3$ can be parametrized by a fct. $f(s,t)$, with $f \in C(\bar{U}, \mathbb{R}^3)$, $U \subset \mathbb{R}^2$ a domain. Then $M = \underbrace{\text{range of } f}_{:= \{y \in \mathbb{R}^3 :\, f(s,t) = y \text{ for some } (s,t) \in \bar{U}\}}$.

We call $\boxed{M \text{ smooth if } f \in C^1(U, \mathbb{R}^3) \text{ and } n := \frac{\partial f}{\partial s} \times \frac{\partial f}{\partial t} \neq 0 \text{ on } U.}$



$\rightarrow$ $\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}$ are tangent vectors (spanning the tangent plane)

$\rightarrow$ n is a normal vector (perpendicular to tangent plane)

We define the unit normal vector $\hat{n} = \frac{n}{\|n\|}$.

Analogous to line integrals, we define:

- the $\boxed{\text{surface area } \sigma(M) := \int_U \|n\|\, dS}$, $\swarrow$ analogous to $l(\gamma) = \int_a^b \|\gamma'(t)\|\, dt$

  $\underbrace{\int_U \|n\|\, dS}_{\text{"Integrating up infinitesimal surface elements"}}$

- for $\Phi \in C(M, \mathbb{R})$ the $\boxed{\text{surface integral } \int_M \Phi \, d\sigma := \int_U \Phi \circ f \, \|n\|\, dS}$, $\swarrow$ analogous to $\int_\gamma \Phi\, ds = \int_a^b \Phi(\gamma(t))\, \|\gamma'(t)\|\, dt$

- for $F \in C(M, \mathbb{R}^3)$ the $\boxed{\text{flux integral } \int_M F \cdot \hat{n}\, d\sigma := \int_U (F \circ f) \cdot n \, dS}$. $\swarrow$ analogous to $\int_\gamma F \cdot dx = \int_a^b (F \circ \gamma)(t) \cdot \gamma'(t)\, dt$

  $\hat{n}$ $\downarrow$ normalized normal vector

Examples:

- Surface area of a sphere: We can choose $f(\theta,\varphi) = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$, $\varphi \in [0,2\pi]$, $\theta \in [0,\pi]$.

(as for spherical coordinates)

Then: $\dfrac{\partial f}{\partial \theta} = \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix}$, $\dfrac{\partial f}{\partial \varphi} = \begin{pmatrix} -\sin\theta\sin\varphi \\ \sin\theta\cos\varphi \\ 0 \end{pmatrix}$

$$\Rightarrow n = \frac{\partial f}{\partial \theta} \times \frac{\partial f}{\partial \varphi} = \begin{pmatrix} 0 + \sin^2\theta\cos\varphi \\ \sin^2\theta\sin\varphi - 0 \\ \cos\theta\sin\theta(\cos^2\varphi + \sin^2\varphi) \end{pmatrix} = \begin{pmatrix} \sin^2\theta\cos\varphi \\ \sin^2\theta\sin\varphi \\ \cos\theta\sin\theta \end{pmatrix}$$

$$\Rightarrow \|n\|^2 = \underbrace{\sin^4\theta\cos^2\varphi + \sin^4\theta\sin^2\varphi}_{=\sin^4\theta} + \cos^2\theta\sin^2\theta = \sin^2\theta(\sin^2\theta + \cos^2\theta) = \sin^2\theta$$

$$\Rightarrow \|n\| = \sin\theta$$

$$\Rightarrow \sigma(\text{sphere}) = \int_0^{2\pi}\int_0^{\pi} \sin\theta \, d\theta \, d\varphi = 2\pi \, (-\cos\theta)\Big|_0^{\pi} = 4\pi.$$

- M = upper hemisphere of radius 1 centered at 0, $\phi(x,y,z) = (x^2+y^2)z$.

We use the same f as above with $\varphi \in [0,2\pi]$ but $\theta \in [0,\frac{\pi}{2}]$ only.

$$\begin{aligned} \Rightarrow \int_M \phi \, d\sigma &= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \phi(f(\theta,\varphi)) \, \|n(\theta,\varphi)\| \, d\theta \, d\varphi \\ &= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \sin^2\theta(\cos^2\varphi + \sin^2\varphi)\cos\theta \, \overbrace{\sin\theta}^{=\|n\|} \, d\theta \, d\varphi \\ &= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \sin^3\theta\cos\theta \, d\theta \, d\varphi \\ &= 2\pi \, \frac{1}{4}\sin^4\theta\Big|_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{2}. \end{aligned}$$

- Same M, $F = \frac{1}{x^2+y^2+z^2}\begin{pmatrix}1\\1\\1\end{pmatrix} \Rightarrow F(f(\theta,\varphi)) = \begin{pmatrix}1\\1\\1\end{pmatrix}$

, since $(\sin\theta\cos\varphi)^2 + (\sin\theta\sin\varphi)^2 + (\cos\theta)^2 = 1$.

$$\Rightarrow \int_M F \cdot \underbrace{\hat{n}}_{\text{outward pointing}} \, d\sigma = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \underbrace{F(f(\theta,\varphi))}_{=\begin{pmatrix}1\\1\\1\end{pmatrix}} \cdot \underbrace{n(\theta,\varphi)}_{=\begin{pmatrix}\sin^2\theta\cos\varphi\\ \sin^2\theta\sin\varphi\\ \cos\theta\sin\theta\end{pmatrix}} \, d\theta \, d\varphi$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \left(\sin^2\theta\cos\varphi + \sin^2\theta\sin\varphi + \cos\theta\sin\theta\right) d\theta \, d\varphi$$

$$= \int_0^{\frac{\pi}{2}} \sin^2\theta \, d\theta \underbrace{\int_0^{2\pi} (\cos\varphi + \sin\varphi) \, d\varphi}_{=0} + 2\pi \left. \frac{1}{2}\sin^2\theta \right|_0^{\frac{\pi}{2}}$$

$$= \pi$$