Recall: we define a surface M by a fct. $f \in C^1(U, \mathbb{R}^3)$ ($U \subset \mathbb{R}^2$ a domain) with normal vector $n := \frac{\partial f}{\partial s} \times \frac{\partial f}{\partial t} \neq 0$. Then we define:

- the surface area $\sigma(M) := \int_U \|n\| \, dS$,
- for $\Phi \in C(M, \mathbb{R})$, the surface integral $\int_M \Phi \, d\sigma := \int_U \Phi \circ f \, \|n\| \, dS$,
- for $F \in C(M, \mathbb{R}^3)$, the flux integral $\int_M F \cdot \hat{n} \, d\sigma := \int_U (F \circ f) \cdot n \, dS$.

Next, we relate flux integrals to volume integrals ("Green's theorem one dimension higher")

3.6 Divergence Theorem

Let us state and discuss the result; we omit proofs.

The divergence thm. holds in any dimension:

Theorem (Divergence Theorem; also called Gauß' Theorem):

Let $D \subset \mathbb{R}^n$ be a domain and let $V \subset D$ s.t.

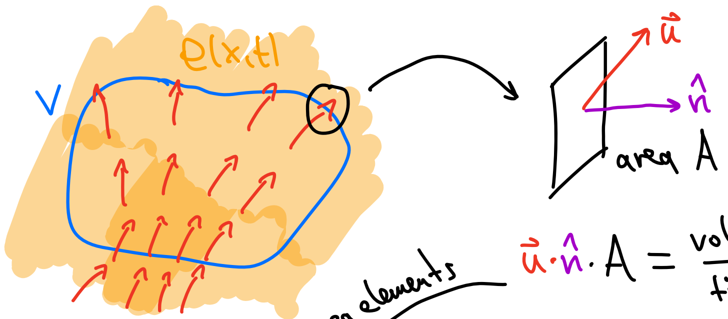
- $\overline{V} \subset D$, and $\overline{V}$ bounded and regular,
- ∂V has non-vanishing piece-wise continuous normal field n,
- $F \in C^1(V, \mathbb{R}^n)$.

Then $\boxed{\int_{\partial V} F \cdot \hat{n} \, d\sigma = \int_V \underbrace{\nabla \cdot F}_{= \operatorname{div} F \text{ ("divergence of } F\text{")}} \, dx}$, with $\hat{n}$ the outward pointing unit normal vector.

Some interpretation / connection to physics:

Let

- $\rho(x,t)$ = density at point x at time t (e.g. mass density or probability density).
- $m_V(t) = \int_V \rho(x,t)\, d^3x$ = total mass in domain $V \subset \mathbb{R}^3$.
- $\vec{j}(x,t)$ = flux density $\left(\text{units: } \frac{1}{\text{time}\cdot\text{area}}\right) = \rho(x,t)\cdot\vec{u}(x,t)$, with $\vec{u}(x,t)$ the velocity vector field.



$\vec{u}\cdot\hat{n}\cdot A = \frac{\text{volume}}{\text{time}}$ passing through surface

integrate up small area elements

Change of mass in time is $\frac{dm_V(t)}{dt} = -\int_{\partial V} \vec{j}\cdot\hat{n}\, d\sigma \overset{\text{Gauß}}{=} -\int_V \operatorname{div}\vec{j}\, d^3x$

by def. of $m_V(t)$ $\quad = \int_V \frac{\partial \rho(x,t)}{\partial t}\, d^3x$

=> Since this holds $\forall$ domains V, we have deduced the

$$\boxed{\text{continuity equation } \frac{\partial\rho}{\partial t} + \operatorname{div}\vec{j} = 0.}$$

For heat ("Fourier's law") or other diffusion processes ("Fick's law") we have

$\vec{j} = -\vec{\nabla}\rho$.

=> $\operatorname{div}\vec{j} = -\vec{\nabla}\cdot\vec{\nabla}\rho = -\underbrace{\Delta}_{\vec{\nabla}\cdot\vec{\nabla} = \frac{\partial^2}{\partial x_1^2} + \dots \frac{\partial^2}{\partial x_n^2} = \text{Laplace operator}}\rho$,

which leads to the $\boxed{\text{heat eq. or diffusion eq. } \frac{\partial\rho}{\partial t} = \Delta\rho.}$

Another result is a generalization of Green's thm. to our surfaces M.

Theorem (Stokes' theorem):
Let $D \subset \mathbb{R}^3$ be a domain, let $M \subset D$ a smooth surface that is bounded and orientable, and let ∂M have smooth parametrization with orientation anti-clockwise w.r.t. to the normal field of M. Let $F \in C^1(D, \mathbb{R}^3)$.

Then $$\boxed{\int_{\partial M} F \cdot dx = \int_M \underbrace{(\nabla \times F)}_{=:\operatorname{curl} F} \cdot \hat{n} \, d\sigma}.$$

Note:
- "Orientable" means that a normal vector field can be chosen consistently (always the case for $M := \operatorname{range} f$).
- For flat surfaces we recover Green's thm.: set $F = \begin{pmatrix} F_1 \\ F_2 \\ 0 \end{pmatrix}$, $\hat{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Before we discuss examples, a few interesting implications.

Corollary: With the same notation as in Stokes' thm., assume that M has no boundary. Then $\int_M (\nabla \times F) \cdot \hat{n} \, d\sigma = 0$ for any $F \in C^1(D, \mathbb{R}^3)$.

Corollary: Let $F \in C^1(D, \mathbb{R}^3)$, $D \subset \mathbb{R}^3$ a simply connected domain. Then

$$F \text{ conservative} \iff \nabla \times F = 0.$$

Sketch of proof: "$\Rightarrow$" If $F = \nabla\phi$, then $\nabla \times F = \nabla \times \nabla \phi = 0$, see homework.

"$\Leftarrow$" Let γ be a closed curve. Under the stated assumptions one can show that there is a "capping surface" M s.t. $\partial M = \gamma$.

Then $\int_\gamma F \cdot dx = \int_M (\nabla \times F) \cdot \hat{n}\, d\sigma = 0 \Rightarrow F$ conservative. $\square$

Examples:

- Let $F(x,y,z) = (x^3, y^3, z^3)$. We compute the flux $\Phi_M(F)$ through the upper hemisphere $M := \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = R^2, z \geq 0\}$ with radius $R > 0$.

$M' := \{(x,y,0) \in \mathbb{R}^3 : x^2+y^2 \leq R^2\}$ the "bottom"

M and M' enclose the volume $V = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq R^2, z \geq 0\}$.

Then $\Phi_M(F) := \int_M F \cdot \hat{n}\, d\sigma = \int_{M \cup M'} F \cdot \hat{n}\, d\sigma - \int_{M'} \underbrace{F \cdot \hat{n}}_{= F \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}\big|_{z=0} = -F_3|_{z=0} = 0} d\sigma$

$$\overset{\text{Gauss}}{=} \int_V \operatorname{div} F \, d^3x$$

$$= 3 \int_V (x^2+y^2+z^2)\, d^3x$$

$$\overset{\text{spherical coordinates}}{=} 3 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^R r^2 \, r^2 \sin\theta \, dr\, d\theta\, d\varphi$$

$$= 3 \cdot 2\pi \cdot 1 \cdot \frac{R^5}{5}$$

$$= \frac{6\pi}{5} R^5.$$

- Example for Stokes' thm.: see homework

Another application: Maxwell's equations in differential and integral form

- $V \subset \mathbb{R}^3$ a bounded volume with closed boundary ∂V
- $M \subset \mathbb{R}^3$ a surface with closed boundary curve ∂M
- ρ = charge density, $Q = \int_V \rho \, d^3x$ the total electric charge within V
- J = el. current density, $I = \int_M J \cdot \hat{n} \, d\sigma$ the el. current passing through M
- E = electric field, B = magnetic field
- c = speed of light

Differential version		Integral version
$\nabla \cdot E = 4\pi\rho$	$\xleftrightarrow{\text{Gauß}}$	$\int_{\partial V} E \cdot \hat{n} \, d\sigma = 4\pi Q$
$\nabla \cdot B = 0$	$\xleftrightarrow{\text{Gauß}}$	$\int_{\partial V} B \cdot \hat{n} \, d\sigma = 0$
$\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t}$	$\xleftrightarrow{\text{Stokes}}$	$\int_{\partial M} E \cdot dx = -\frac{1}{c}\frac{d}{dt}\int_M B \cdot \hat{n} \, d\sigma$
$\nabla \times B = \frac{1}{c}\left(4\pi J + \frac{\partial E}{\partial t}\right)$	$\xleftrightarrow{\text{Stokes}}$	$\int_{\partial M} B \cdot dx = \frac{1}{c}\left(4\pi I + \frac{d}{dt}\int_M E \cdot d\sigma\right)$