Advanced Calculus and Methods of Mathematical Physics — Session 26

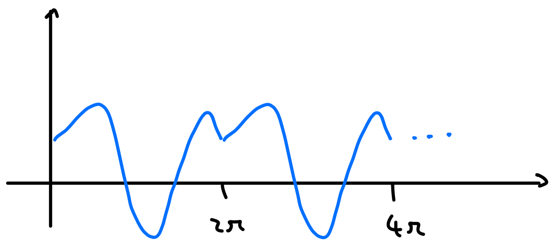
Prof. Sören Petrat, Jacobs University, Spring 2022 — May 6, 2022

---

4. Fourier Series

We consider 2π-periodic functions, i.e., $f(x+2\pi)=f(x)$

(L-periodic for any $0 \neq L \in \mathbb{R}$ works analogously).



Fourier series: • idea: decompose functions into "pure frequencies" (e.g., signals)

• works also for non-differentiable functions (as opposed to Taylor series)

Let us just consider one period, i.e., $f: [0,2\pi] \to \mathbb{C}$, $f(0)=f(2\pi)$.

We assume f is Riemann-integrable on $[0,2\pi]$.

Then the $\boxed{\text{Fourier series of } f \text{ is defined as } F_f(x) := \sum_{k=-\infty}^{\infty} \underbrace{\hat{f}_k}_{=\text{Fourier coefficients}} \overbrace{e^{ikx}}^{=\cos kx + i \sin kx}}$.

Note: $e_k(x) := e^{ikx}$ plays the role of a basis function.

Let us introduce the inner product $\langle f,g\rangle := \frac{1}{2\pi}\int_0^{2\pi} \overline{f(x)}\, g(x)\, dx$ and norm $\|f\| = \sqrt{\langle f,f\rangle}$.

Then $\langle e_j, e_k\rangle = \frac{1}{2\pi}\int_0^{2\pi} \overline{e_j(x)}\, e_k(x)\, dx = \frac{1}{2\pi}\int_0^{2\pi} e^{-ijx} e^{ikx}\, dx = \frac{1}{2\pi}\begin{cases} \frac{1}{i(k-j)} e^{i(k-j)x}\Big|_0^{2\pi} = 0 & k \neq j \\ 2\pi & k=j \end{cases}$

$= \underbrace{\delta_{jk}}_{\text{Kronecker delta}} = \begin{cases} 1 & \text{if } j=k \\ 0 & \text{if } j \neq k \end{cases}$

Now assuming $F_f(x)$ converges uniformly to $f(x)$, we have

$$\langle e_j, f\rangle = \langle e_j, \sum_{k=-\infty}^{\infty} \hat{f}_k e_k\rangle \underset{\substack{\uparrow \\ \text{uniform} \\ \text{convergence}}}{=} \sum_{k=-\infty}^{\infty} \hat{f}_k \underbrace{\langle e_j, e_k\rangle}_{\delta_{jk}} = \hat{f}_j .$$

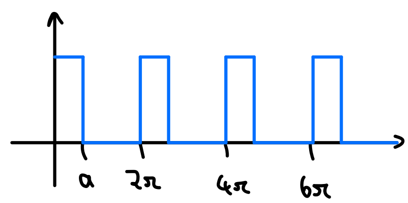
So far we know: If $f(x) := \sum_{k=-\infty}^{\infty} \hat{f}_k e^{ikx}$ is uniformly convergent, then $\hat{f}_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikx} f(x)\,dx$.

But we can define $\hat{f}_k$ for any Riemann integrable f.

So generally, we define the $\boxed{\text{Fourier transform of } f \text{ as } \hat{f}_k := \langle e_k, f\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikx} f(x)\,dx.}$

Question: Does $\sum_{k=-\infty}^{\infty} \hat{f}_k e^{ikx}$ always converge to $f(x)$, and if yes, in what sense?

Example A: $f(x) = \begin{cases} 1 & \text{for } x \in [0,a) \\ 0 & \text{for } x \in [a, 2\pi) \end{cases}$



We find $\cdot\ \hat{f}_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{a}{2\pi}$

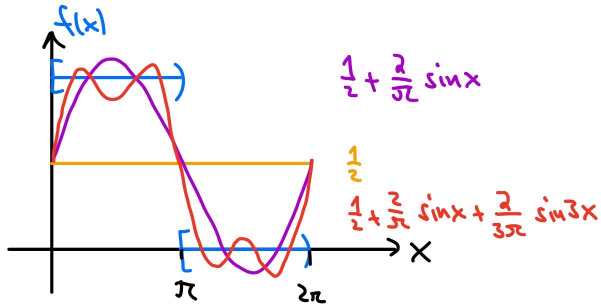
$\cdot$ For $k \neq 0$: $\hat{f}_k = \frac{1}{2\pi}\int_0^{a} e^{-ikx} f(x)\,dx = \frac{1}{2\pi}\int_0^{a} e^{-ikx}\,dx = \frac{1}{2\pi}\frac{1}{(-ik)} e^{-ikx}\Big|_0^a$

$$= \frac{i}{2\pi k}\left(e^{-ika} - 1\right)$$

E.g., for $a = \pi$, we have $\hat{f}_k = \frac{i}{2\pi k}\left(e^{-i\pi k} - 1\right) = \frac{i}{2\pi k}\left((-1)^k - 1\right) = \begin{cases} 0 & \text{for } k \text{ even} \\ \frac{-i}{\pi k} & \text{for } k \text{ odd} \end{cases}$

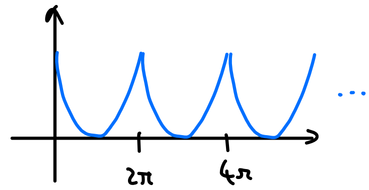
and $F_f(x) = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{ikx} = \underbrace{\frac{1}{2}}_{\hat{f}_0} + \sum_{\substack{k=1 \\ k\text{ odd}}}^{\infty} \frac{(-i)}{\pi k} e^{ikx} + \underbrace{\sum_{\substack{k=-\infty \\ k\text{ odd}}}^{-1} \frac{(-i)}{\pi k} e^{ikx}}_{= -\sum_{\substack{k=1 \\ k\text{ odd}}}^{\infty} \frac{(-i)}{\pi k} e^{-ikx}} = \frac{1}{2} + \sum_{\substack{k=1 \\ k\text{ odd}}}^{\infty} \frac{(-i)}{\pi k} \underbrace{\left(e^{ikx} - e^{-ikx}\right)}_{\substack{i\sin kx - (-i\sin kx) \\ = 2i\sin kx}},$

i.e., $F_f(x) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{2}{\pi k} \sin kx$.



Here, e.g., we see that $F_f(\pi) = \frac{1}{2} \neq f(\pi)$, so we have neither pointwise nor uniform convergence. But it looks like some type of convergence should hold.

Example B: $f(x) = (x-\pi)^2$ on $[0, 2\pi]$



A computation (see HW) shows $F_f(x) = \frac{\pi^2}{3} + \sum_{k \neq 0} \frac{2}{k^2} e^{ikx} = \frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4}{k^2} \cos kx$,

which converges uniformly (according to the Weierstrass M-test), i.e., $F_f(x) = f(x)$.

As a corollary we find $\sum_{k=1}^{\infty} \frac{4}{k^2} = f(0) - \frac{\pi^2}{3} = \pi^2 - \frac{\pi^2}{3} = \frac{2\pi^2}{3}$, i.e., $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

Question: What is the right kind of convergence for functions as in Example A?

Answer: Convergence in the norm coming from our inner product.

First, note that

$$\|f - \sum_{k=-n}^{n} \hat{f}_k e_k\|^2 = \langle f - \sum_{k=-n}^{n} \hat{f}_k e_k, f - \sum_{k=-n}^{n} \hat{f}_k e_k \rangle$$

$$= \|f\|^2 - \sum_{k=-n}^{n} \Big(\underbrace{\langle f, \hat{f}_k e_k \rangle}_{\substack{= \overline{\hat{f}_k} \langle f, e_k \rangle \\ = \overline{\hat{f}_k} \hat{f}(k) = |\hat{f}_k|^2}} + \underbrace{\langle \hat{f}_k e_k, f \rangle}_{= \hat{f}_k \overline{\hat{f}_k} = |\hat{f}_k|^2} \Big) + \sum_{k=-n}^{n} \sum_{j=-n}^{n} \underbrace{\langle \hat{f}_j e_j, \hat{f}_k e_k \rangle}_{= \hat{f}_j \overline{\hat{f}_k} \delta_{jk}}$$

$$\Rightarrow \left\|f - \sum_{k=-n}^{n} \hat{f}_k e_k\right\|^2 = \|f\|^2 - \sum_{k=-n}^{n} |\hat{f}_k|^2$$

As a corollary, we get $\boxed{\text{Bessel's inequality } \sum_{k=-\infty}^{\infty} |\hat{f}_k|^2 \le \|f\|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2\,dx}$.

Furthermore: $\underbrace{\left\|f - \sum_{k=-n}^{n} \hat{f}_k e_k\right\| \xrightarrow{n\to\infty} 0}_{\text{called "mean-square convergence"}} \iff \|f\|^2 = \sum_{k=-\infty}^{\infty} |\hat{f}_k|^2$ (Parseval identity)

For Example A we find $\|f\|^2 = \frac{1}{2\pi}\int_0^a dx = \frac{a}{2\pi}$ and

$$\sum_{k=-\infty}^{\infty} |\hat{f}_k|^2 = \underbrace{\left(\frac{a}{2\pi}\right)^2}_{=|\hat{f}_0|^2} + \sum_{k\neq 0} \Bigg|\underbrace{\frac{i}{2\pi k}\left(e^{-ika} - 1\right)}_{\hat{f}_k}\Bigg|^2$$

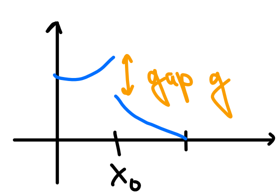
$= \ldots$ (see HW; use results from Ex. B)

$= \frac{a}{2\pi}$, i.e., the Fourier series converges to f in mean-square.

In general, we can approximate any Riemann-integrable f by such square pulses, which leads to the following result:

$\boxed{\begin{array}{l}\underline{\text{Theorem:}} \text{ Let } f:[0,2\pi]\to\mathbb{C},\ f(0)=f(2\pi) \text{ be Riemann-integrable.} \\ \text{Then } \left\|f - \sum_{k=-n}^{n} \hat{f}_k e^{ikx}\right\| \xrightarrow{n\to\infty} 0, \text{ i.e., } \sum_{k=-\infty}^{\infty} \hat{f}_k e^{ikx} \longrightarrow f(x) \text{ in mean-square.}\end{array}}$

Let us mention two more properties of the Fourier series. Suppose f is piece-wise continuous and piece-wise differentiable but has a discontinuity at x_0:



Then:

- $\sum_{k=-n}^{n} \hat{f}_k e^{ikx} \xrightarrow{n\to\infty} \frac{1}{2}\Big(\underbrace{\lim_{x \downarrow 0} f(x)}_{=: f(x_0^+)} + \underbrace{\lim_{x \uparrow 0} f(x)}_{=: f(x_0^-)} \Big)$ (as we saw in Example A)

- Let $g := f(x_0^+) - f(x_0^-)$ be the gap at the discontinuity.

Then $\sum_{k=-n}^{n} \hat{f}_k e^{ikx} \Big|_{x = x_0 + \frac{\pi}{n}} \xrightarrow{n\to\infty} f(x_0^+) + g\,c$, with $c \approx 0.089\ldots$

and $\sum_{k=-n}^{n} \hat{f}_k e^{ikx} \Big|_{x = x_0 - \frac{\pi}{n}} \xrightarrow{n\to\infty} f(x_0^-) - g\,c$.

$\Rightarrow$ Near a discontinuity, the Fourier series is ~ 9% off. This is called "Gibbs phenomenon".