5. Complex Analysis

In this chapter we consider functions $f: D \to \mathbb{C}$, where $D \subset \mathbb{C}$ is a domain. This will lead to a deeper understanding of functions such as exp and log (hence the German name "Funktionentheorie") and also useful tools, e.g., for integration.

Differentiability is defined in the usual way:

A fct. $f: D \to \mathbb{C}$ ($D \subset \mathbb{C}$ a domain) is differentiable at $z_0 \in D$ if

$$f(z) = f(z_0) + c(z-z_0) + |z-z_0|\, h(z) \quad \text{for some } c \in \mathbb{C} \text{ and with } \lim_{z\to z_0} h(z) \to 0.$$

Alternatively, we can identify $\mathbb{C}$ with $\mathbb{R}^2$ by writing $z = x+iy = \begin{pmatrix} x \\ y \end{pmatrix}$ and $f = u+iv = \begin{pmatrix} u \\ v \end{pmatrix}$.

$f: D \subset \mathbb{R}^2 \to \mathbb{R}^2,\ \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$ is differentiable at $z_0 = x_0 + iy_0$ if

$$f(z) = f(z_0) + A\cdot(z-z_0) + |z-z_0|\, h(z) \quad \text{for some real } 2\times 2 \text{ matrix } A, \text{ and } \lim_{z\to z_0} h(z) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

If f is differentiable, then $A = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$.

Now we can compare both linear expressions (let us set $z_0 = 0$ here for simplicity):

$$cz = (a+ib)(x+iy) = (ax-by) + i(bx+ay) = \begin{pmatrix} ax-by \\ bx+ay \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \overset{!}{=} A \begin{pmatrix} x \\ y \end{pmatrix}.$$

Comparing with A yields: $\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \text{ the Cauchy-Riemann equations.}}$

("C-R eq.s")

We have proven

$\boxed{\text{Theorem: A function } f: D \to \mathbb{C},\ D \subset \mathbb{C} \text{ a domain, is complex differentiable ("holomorphic") at } z_0 \in D \text{ if and only if } f \text{ is real differentiable and the C-R eq.s hold at } z_0.}$

Remark: The C-R eq.s have many interesting consequences and make holomorphic fct.s very interesting.

Examples: check that z^n, e^z are holomorphic, but $\bar{z}$ is not.