Advanced Calculus and Methods of Mathematical Physics — Session 28

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Last time we have proven:

> **Theorem:** A function $f: D \to \mathbb{C}$, $D \subset \mathbb{C}$ a domain, is complex differentiable ("holomorphic") at $z_0 \in D$ if and only if f is real differentiable and the C-R eq.s hold at z_0.

Next, consider complex line integrals. Let γ be a curve in D. Then

$$\int_\gamma f(z)\,dz = \int_\gamma (u+iv)(dx+idy) = \int_\gamma (u\,dx - v\,dy) + i\int_\gamma (v\,dx + u\,dy)$$

$$= \int_\gamma \begin{pmatrix} u \\ -v \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \end{pmatrix} + i \int_\gamma \begin{pmatrix} v \\ u \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \end{pmatrix}.$$

Now suppose γ is simple and closed, and $\gamma = \partial S$ for some $S \subset D$.

Then
$$\int_\gamma f(z)\,dz \overset{\text{Green's thm.}}{=} \int_S \underbrace{\nabla^\perp \cdot \begin{pmatrix} u \\ -v \end{pmatrix}}_{= \begin{pmatrix} -\frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} \end{pmatrix} \cdot \begin{pmatrix} u \\ -v \end{pmatrix} = -\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \underset{\text{C-R eq.s}}{=} 0} dx\,dy + i \int_S \underbrace{\nabla^\perp \cdot \begin{pmatrix} v \\ u \end{pmatrix}}_{= \begin{pmatrix} -\frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} \end{pmatrix} \cdot \begin{pmatrix} v \\ u \end{pmatrix} = -\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} \underset{\text{C-R eq.s}}{=} 0} dx\,dy = 0$$

We have proven Cauchy's integral theorem:

This is the most general condition for Green's thm. to hold.

> **Theorem:** Let $f: D \to \mathbb{C}$, $D \subset \mathbb{C}$ a simply connected domain, be holomorphic, and $\gamma \subset D$ a closed curve. Then $\int_\gamma f(z)\,dz = 0$.

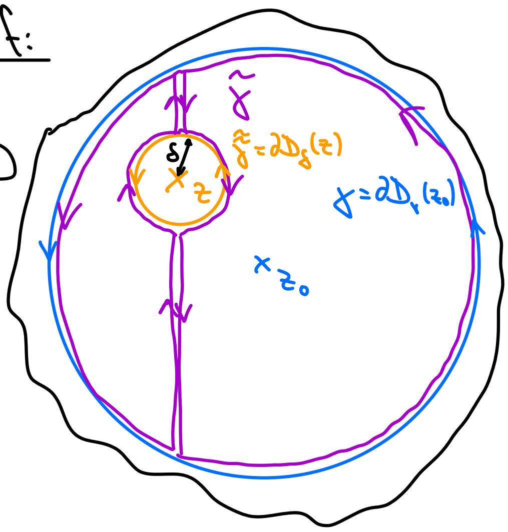
This theorem implies Cauchy's integral formula:

Corollary: Let $f: D \to \mathbb{C}$, $D \subset \mathbb{C}$ a domain, be holomorphic. Then

$$f(z) = \frac{1}{2\pi i} \int_{\partial D_r(z_0)} \frac{f(w)}{w-z}\, dw \quad \text{for any } z \in \underbrace{D_r(z_0)}_{\text{disc with radius } r \text{ around } z_0} := \{w \in \mathbb{C} : |z_0 - w| \le r\} \subset D$$

↑ counter clockwise

Proof:



We have: $\displaystyle\int_\gamma \frac{f(w)}{w-z}\,dw = \underbrace{\int_{\tilde{\gamma}} \frac{f(w)}{w-z}\,dw}_{} + \int_{\tilde{\tilde{\gamma}}} \frac{f(w)}{w-z}\,dw$

$$= \int_{\tilde{\gamma}_R} \frac{f(w)}{w-z}\,dw + \int_{\tilde{\gamma}_L} \frac{f(w)}{w-z}\,dw \overset{\text{by Cauchy's int. thm.}}{=} 0.$$

holomorphic in the enclosed areas (which are simply connected, even star-shaped for δ small enough)

$$\Rightarrow \text{For any } \delta > 0: \int_{\partial D_r(z_0)} \frac{f(w)}{w-z}\,dw = \int_{\partial D_\delta(z)} \frac{f(w)}{w-z}\,dw \overset{\substack{\text{shift (change of}\\ \text{variables) } w-z \to z}}{=} \frac{1}{2\pi i}\int_{\partial D_\delta(0)} \frac{f(w+z)}{w}\,dw$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z+\gamma(t))}{\gamma(t)}\gamma'(t)\,dt \overset{\gamma:[0,2\pi]\to\mathbb{C},\ \gamma(t)=\delta e^{it}}{=} \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z+\delta e^{it})}{\delta e^{it}}\, i\delta e^{it}\,dt = \frac{1}{2\pi}\int_0^{2\pi} f(z+\delta e^{it})\,dt$$

by uniform convergence! $\xrightarrow{\delta \to 0} f(z)$. $\square$

Note: the proof clearly works as well if we replace $\partial D_r(z_0)$ by any closed simple curve γ encircling z.

It follows:

$\boxed{\text{\underline{Corollary}: Under the same assumptions as above, } f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(w)}{(w-z)^{n+1}}\,dw.}$

$\boxed{\text{\underline{Corollary}: If } f: D \to \mathbb{C},\ D \subset \mathbb{C} \text{ a domain, is holomorphic, then } f \in C^\infty \text{ and furthermore } f \text{ is analytic, i.e., it has a convergent Taylor series.}}$

<u>Proof:</u> $\frac{1}{w-z} = \frac{1}{w-z_0+z_0-z} = \frac{1}{w-z_0}\,\frac{1}{1-\frac{z-z_0}{w-z_0}} = \frac{1}{w-z_0}\sum_{k=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^k$ for $\left|\frac{z-z_0}{w-z_0}\right| < 1$,

geometric series with uniform convergence!

$$\Rightarrow f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw = \sum_{k=0}^{\infty}(z-z_0)^k\,\underbrace{\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)^{k+1}}\,dw}_{=\frac{f^{(k)}(z_0)}{k!}} = \sum_{k=0}^{\infty}\frac{f^{(k)}(z_0)}{k!}(z-z_0)^k. \quad \square$$

take sum out of integral by uniform convergence

Next: More generally, one can write down a $\boxed{\text{Laurent series } \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k}$.

It converges if $\cdot\ |z-z_0| < R$ (= radius of convergence of $\sum_{k=0}^{\infty} a_k (z-z_0)^k$),

$\cdot$ and $\frac{1}{|z-z_0|} < \tilde{R}$ (= radius of convergence of $\sum_{k=-\infty}^{-1} a_k (z-z_0)^k$),

i.e., it converges on an annulus $\Omega = \{z \in \mathbb{C} : \frac{1}{\tilde{R}} < |z-z_0| < R\}$.

Indeed, one can show that any holomorphic function $f: \Omega \to \mathbb{C}$ has a Laurent series.

An interesting case is when $\tilde{R} = \infty$, i.e., $0 < |z-z_0| < R$. Such points z_0 are called $\boxed{\text{isolated singularities}}$.

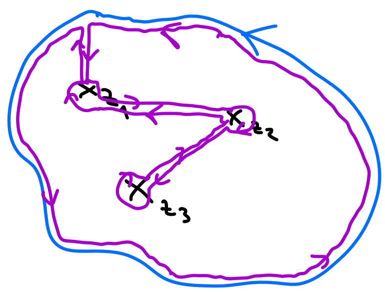
For these, we get:

$$\int_{\partial D_r(z_0)} f(z)\,dz = \sum_{k=-\infty}^{\infty} a_k \underbrace{\int_{\partial D_r(z_0)} (z-z_0)^k\,dz}_{} = 2\pi i\, a_{-1}.$$

$$= \int_0^{2\pi} (re^{it})^k\, ire^{it}\,dt = ir^{k+1}\int_0^{2\pi} e^{i(k+1)t}\,dt = 2\pi i r^{k+1}\delta_{k,-1}$$

We call $a_{-1} := \mathrm{Res}(f, z_0)$ the residue of f at z_0.

Generalizing along the picture



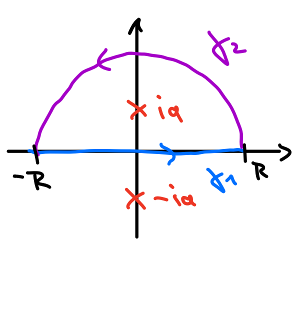
yields:

Theorem (Residue Theorem): Let $D \subset \mathbb{C}$ be a simply connected domain. Let $f: D \to \mathbb{C}$ be holomorphic except at a finite number of isolated points $z_1, \dots, z_n$. Let γ be a simple closed curve enclosing all $z_1, \dots, z_n$. Then

$$\boxed{\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f, z_k).}$$

Application: Compute $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^4}\,dx$, $0 \neq a \in \mathbb{R}$.

We def. $f(z) = \frac{1}{(z^2+a^2)^4}$. It has two isolated singularities at $\pm ia$.



The residue thm. tells us that for R large enough

$$\underbrace{\int_{\gamma_1} f(z)\,dz}_{= \int_{-R}^{R} \frac{1}{(x^2+a^2)^4}dx} + \underbrace{\int_{\gamma_2} f(z)\,dz}_{\overset{\curvearrowleft}{=} \int_0^{\pi} (R^2e^{2it}+a^2)^{-4}\, iRe^{it}\,dt} = \int_{\gamma=\gamma_1+\gamma_2} f(z)\,dz = 2\pi i\,\mathrm{Res}(f, ia).$$

Note that $\left|\int_{\gamma_2} f(z)\,dz\right| \le \int_0^{\pi} |R^2e^{2it}+a^2|^{-4}\, R\,dt \le \frac{R}{|R^2-a^2|^4}\underbrace{\int_0^{\pi} dt}_{=\pi} \xrightarrow{R\to\infty} 0.$

Thus, we just need to compute $\operatorname{Res}(f,ia)$. Let us write $z = ia+w$ and

$$f(z) = \left((ia+w)^2+a^2\right)^{-4} = \left(-a^2+2iaw+w^2+a^2\right)^{-4} = w^{-4}\left(2ia+w\right)^{-4}$$

$$= (2iaw)^{-4}\underbrace{\left(1+\frac{w}{2ia}\right)^{-4}}_{=\sum_{k=0}^{\infty}\binom{-4}{k}\left(\frac{w}{2ia}\right)^k}$$

$\operatorname{Res}(f,ia)$ is the coefficient with power w^{-1}, i.e. $k=3$. Then

$$\operatorname{Res}(f,ia) = (2ia)^{-4}\binom{-4}{3}\left(\frac{1}{2ia}\right)^3 = (2ia)^{-7}\,\frac{(-4)(-4-1)(-4-2)}{3!} = \dots = -i\,\frac{5}{32}\,a^7.$$

$$\Longrightarrow \int_{-\infty}^{\infty}\frac{1}{(x^2+a^2)^4}\,dx = \lim_{R\to\infty}\int_{-R}^{R}\frac{1}{(x^2+a^2)^4}\,dx = 2\pi i\left(-i\,\frac{5}{32}a^7\right) = \pi\,\frac{5}{16}\,a^7.$$