Jacobs University
Fall 2021

December 22, 2021

Foundations of Mathematical Physics

Final Exam

Instructions:

- Do all the work on this exam paper.
- Show your work, i.e., carefully write down the steps of your solution. You will receive points not just based on your final answer, but on the correct steps in your solution.
- No tools or other resources are allowed for this exam. In particular, no notes and no calculators.
- You are free to refer to any results proven in class or the homework sheets unless stated otherwise (and unless the problem is to reproduce a result from class or the homework sheets).

Name: Solutions

Problem 1: Schwartz Space and Dilations [25 points]

Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space as defined in class. For $p \in [1, \infty)$ and $\sigma > 0$ we define the L^p dilation with σ as $D^p_\sigma : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d), f(x) \mapsto (D^p_\sigma f)(x) = \sigma^{-d/p} f(x/\sigma)$.

(4) (a) State a clear definition of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$. In particular, define the semi-norms $\|\cdot\|_{\alpha,\beta}$ that we used in the definition.

(4) (b) Is continuity of a function from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$ the same as sequential continuity? Give a brief explanation of your answer (but no complete proof is necessary).

(7) (c) Prove that $D^p_\sigma : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ is continuous.

(5) (d) Show that $\|D^p_\sigma f\|_{L^p(\mathbb{R}^d)} = \|f\|_{L^p(\mathbb{R}^d)}$.

(5) (e) Compute $\mathcal{F} D^2_\sigma f$ and give a brief interpretation of the result.
(Recall that the Fourier transform $\mathcal{F}$ of a function g on Schwartz space is defined as $(\mathcal{F}g)(k) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} g(x) e^{-ikx}\, dx$.)

a) $S(\mathbb{R}^d) = \{ f \in C^\infty(\mathbb{R}^d) : \|f\|_{\alpha,\beta} < \infty \ \forall$ multi-indices $\alpha, \beta \in \mathbb{N}_0^d \}$, (3)

where $\|f\|_{\alpha,\beta} := \sup_{x \in \mathbb{R}^d} |x^\alpha \partial_x^\beta f(x)|$ (1)

b) Yes, (2) because $S(\mathbb{R}^d)$ is a metric space, (2) and on metric spaces continuity and sequential continuity are the same.

c) We assume $(f_n)_{n\in\mathbb{N}}$ is a sequence in $S(\mathbb{R}^d)$ with $\|f_n - f\|_{\alpha,\beta} \xrightarrow{n\to\infty} 0$ for some $f \in S(\mathbb{R}^d)$, for all $\alpha, \beta \in \mathbb{N}_0^d$. (2)

We need to show that $\|D^p_\sigma f_n - D^p_\sigma f\|_{\alpha,\beta} \xrightarrow{n\to\infty} 0$ for all $\alpha, \beta \in \mathbb{N}_0^d$. (1)

$$\|D^p_\sigma f_n - D^p_\sigma f\|_{\alpha,\beta} := \sup_{x\in\mathbb{R}^d} \left| x^\alpha \partial_x^\beta \left(\sigma^{-\frac{d}{p}} f_n\left(\tfrac{x}{\sigma}\right) - \sigma^{-\frac{d}{p}} f\left(\tfrac{x}{\sigma}\right) \right) \right|$$

$$= \sigma^{-\frac{d}{p}} \sigma^{-|\beta|} \sup_{x\in\mathbb{R}^d} \left| x^\alpha \left(f_n^{(\beta)}\left(\tfrac{x}{\sigma}\right) - f^{(\beta)}\left(\tfrac{x}{\sigma}\right) \right) \right|$$

$$= \sigma^{-\frac{d}{p}} \sigma^{-|\beta|} \sigma^{|\alpha|} \sup_{y\in\mathbb{R}^d} \left| y^\alpha \partial_y^\beta \left(f_n(y) - f(y) \right) \right|$$

$$= \sigma^{-\frac{d}{p} - |\beta| + |\alpha|} \|f_n - f\|_{\alpha,\beta} \xrightarrow{n\to\infty} 0 \quad (4)$$

Problem 1: Extra Space

Problem 1: Extra Space

d) We compute

$$\|D_6^p f\|_{L^p}^p := \int_{\mathbb{R}^d} |(D_6^p f)(x)|^p \, dx$$

$$= \int_{\mathbb{R}^d} |6^{-\frac{d}{p}} f(\tfrac{x}{6})|^p \, dx$$

$$= 6^{-d} \int_{\mathbb{R}^d} |f(\tfrac{x}{6})|^p \, dx$$

$$\tfrac{x}{6} =: y \quad = \int_{\mathbb{R}^d} |f(y)|^p \, dy$$

$$= \|f\|_{L^p}^p . \qquad (5)$$

e) $$(\mathcal{F} D_6^2 f)(k) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} (D_6^2 f)(x) e^{-ikx} \, dx$$

$$= (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} 6^{-\frac{d}{2}} f(\tfrac{x}{6}) e^{-ikx} \, dx$$

$$\tfrac{x}{6} := y \quad = (2\pi)^{-\frac{d}{2}} 6^{\frac{d}{2}} \int_{\mathbb{R}^d} f(y) e^{-ik6y} \, dy$$

$$= 6^{\frac{d}{2}} (\mathcal{F} f)(k6)$$

$$= (D_{\frac{1}{6}}^2 \mathcal{F} f)(k)$$

$\Rightarrow \mathcal{F} D_6^2 f = D_{1/6}^2 \mathcal{F} f$, i.e., a dilation with 6 in position space corresponds to a dilation with $\frac{1}{6}$ in Fourier space. (5)

Problem 1: Extra Space

Problem 2: Fourier Transform and the Free Schrödinger Equation [25 points]
Let $\mathcal{F}(f)(k) := (2\pi)^{-d/2}\int_{\mathbb{R}^d} f(x)e^{-ikx}\,dx$ denote the Fourier transform of a function f.

(7) (a) Prove the Plancherel identity, i.e., that

$$\int_{\mathbb{R}^d} \left|(\mathcal{F}f)(k)\right|^2 \mathrm{d}k = \int_{\mathbb{R}^d} \left|f(x)\right|^2 \mathrm{d}x$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$.

(7) (b) Let $\psi_0 \in \mathcal{S}(\mathbb{R}^d)$ and consider the solution to the free Schrödinger equation

$$\psi : \mathbb{R}_t \to \mathcal{S}(\mathbb{R}^d), t \mapsto \psi(t) = \mathcal{F}^{-1}e^{-i\frac{k^2}{2}t}\mathcal{F}\psi_0.$$

Prove that this map is differentiable.

(4) (c) For fixed t, let us consider $\psi(t) \in \mathcal{S}(\mathbb{R}^d)$ as defined in part (b). Prove that its L^2 norm is conserved, i.e., that

$$\|\psi(t)\|_{L^2(\mathbb{R}^d)} = \|\psi_0\|_{L^2(\mathbb{R}^d)}.$$

(3) (d) Let $\mathcal{S}'(\mathbb{R}^d)$ be the space of tempered distributions, i.e., the dual space of $\mathcal{S}(\mathbb{R}^d)$. How is the Fourier transform of $T \in \mathcal{S}'(\mathbb{R}^d)$ defined?

(4) (e) Let $f \in \mathcal{S}(\mathbb{R}^d)$ and define the tempered distribution $T_f \in \mathcal{S}'(\mathbb{R}^d)$ by $T_f(g) = \int_{\mathbb{R}^d} f(x)g(x)\,\mathrm{d}x$ for all $g \in \mathcal{S}(\mathbb{R}^d)$. Compute the Fourier transform of T_f.

a) We compute:

$$\int_{\mathbb{R}^d} |(\mathcal{F}f)(k)|^2 dk := \int_{\mathbb{R}^d} (\mathcal{F}f)(k)\underbrace{\overline{(\mathcal{F}f)(k)}} dk$$

$$= (2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d} \overline{f(x)e^{-ikx}}\,dx = (2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d} \overline{f(x)}e^{ikx}\,dx$$

$$= (\mathcal{F}^{-1}\overline{f})(k) =: g(k)$$

$$\Rightarrow \int_{\mathbb{R}^d} |(\mathcal{F}f)(k)|^2 dk = \int_{\mathbb{R}^d}\left[(2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d} f(x)e^{-ikx}dx\right] g(k)\,dk$$

$$\overset{\text{Fubini}}{=} \int_{\mathbb{R}^d}\left[(2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d} g(k)e^{-ikx}dk\right] f(x)\,dx$$

$$= \int_{\mathbb{R}^d} \underbrace{(\mathcal{F}g)(x)}_{=(\mathcal{F}\mathcal{F}^{-1}\overline{f})(x) = \overline{f(x)}} f(x)\,dx = \int_{\mathbb{R}^d} |f(x)|^2 dx \quad (7)$$

Problem 2: Extra Space

Problem 2: Extra Space

d) For $T \in S'(\mathbb{R}^d)$ we define the Fourier transform $\hat{T} \in S'(\mathbb{R}^d)$ via the formula $\hat{T}(f) := T(\hat{f}) \ \forall f \in S(\mathbb{R}^d)$. (3)

e) For all $g \in S(\mathbb{R}^d)$ we have

$$\hat{T}_f(g) \underset{\text{part d)}}{\uparrow}{:=} T_f(\hat{g}) := \int_{\mathbb{R}^d} f(x)\hat{g}(x)\,dx \underset{\text{Plancherel}}{\uparrow}{=} \int_{\mathbb{R}^d} \hat{f}(x) g(x)\,dx =: T_{\hat{f}}(g),$$

i.e., $\hat{T}_f = T_{\hat{f}}$. (4)

Problem 2: Extra Space (2)

b) Let us define $\dot{\psi}(t,x) := -i\left(\mathcal{F}^{-1}\frac{k^2}{2}e^{-i\frac{k^2}{2}t}\mathcal{F}\psi_0\right)(x)$ (which is in $S(\mathbb{R}^d)$).

Then:

$$\left\|\frac{\psi(t+h)-\psi(t)}{h} - \dot{\psi}(t)\right\|_{\alpha,\beta} \xrightarrow{h\to 0} 0 \quad \forall \alpha,\beta \in \mathbb{N}_0^d \text{ is equivalent to}$$

$$\left\|\mathcal{F}\left(\frac{\psi(t+h)-\psi(t)}{h} - \dot{\psi}(t)\right)\right\|_{\alpha,\beta} \xrightarrow{h\to 0} 0 \quad \forall \alpha,\beta \in \mathbb{N}_0^d \text{ by continuity of } \mathcal{F}. \text{ (2)}$$

We compute:

$$\left\|\mathcal{F}\left(\frac{\psi(t+h)-\psi(t)}{h} - \dot{\psi}(t)\right)\right\|_{\alpha,\beta} = \sup_{k\in\mathbb{R}^d}\left|k^\alpha\partial_k^\beta\left(\frac{e^{-i\frac{k^2}{2}(t+h)}-e^{-i\frac{k^2}{2}t}}{h} + i\frac{k^2}{2}e^{-i\frac{k^2}{2}t}\right)\hat{\psi}_0(k)\right|$$

$\xrightarrow{h\to 0} 0$ since $\hat{\psi}_0(k) \in S(\mathbb{R}^d)$, i.e., $\hat{\psi}_0 \in C^\infty$ in particular, and since $e^{-i\frac{k^2}{2}t}$ is smooth in k and t. (3)

c) $$\|\psi(t)\|_{L^2}^2 := \int_{\mathbb{R}^d}|\psi(t,x)|^2dx = \int_{\mathbb{R}^d}\left|\left(\mathcal{F}^{-1}e^{-i\frac{k^2}{2}t}\mathcal{F}\psi_0\right)(x)\right|^2dx$$

$$\overset{\text{Plancherel}}{=} \int_{\mathbb{R}^d}\left|e^{-i\frac{k^2}{2}t}(\mathcal{F}\psi_0)(k)\right|^2dk$$

$$= \int_{\mathbb{R}^d}|(\mathcal{F}\psi_0)(k)|^2dk$$

$$\overset{\text{Plancherel}}{=} \int_{\mathbb{R}^d}|\psi_0(x)|^2dx =: \|\psi_0\|_{L^2}^2 \quad (4)$$

Problem 3: Bounded Operators [25 points]

(15) (a) Let X and Y be normed spaces, and let $L : X \to Y$ be linear. Prove that L is continuous if and only if it is bounded.

(10) (b) Let $(\varphi_n)_{n\in\mathbb{N}}$ be an orthonormal basis of a Hilbert space $\mathcal{H}$. We define a sequence $(A_n)_{n\in\mathbb{N}}$ of bounded linear operators in $\mathcal{H}$ by

$$A_n\psi := \cancel{\sum_{k=1}^{n} \langle \psi, \varphi_n\rangle \varphi_n} \sum_{k=1}^{n} \langle \varphi_k, \psi\rangle \varphi_k$$

for all $\psi \in \mathcal{H}$, where $\langle\cdot,\cdot\rangle$ is the scalar product on $\mathcal{H}$. Prove that $(A_n)_{n\in\mathbb{N}}$ converges strongly to the identity, but not in operator norm.

a) (i) We prove: L bounded $\Rightarrow$ L continuous.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X with $\|x_n - x\| \to 0$ for some $x \in X$.

Then $\|Lx_n - Lx\| \underset{\uparrow \text{ L linear}}{=} \|L(x_n - x)\| \le C\|x_n - x\| \to 0$, i.e., L is continuous. (5)

(ii) We prove: L continuous $\Rightarrow$ L bounded.

Suppose L not bounded; then there exists a sequence $(x_n)_n$ in X with $\|x_n\| = 1$ $\forall n \in \mathbb{N}$ but $\|Lx_n\| > n$ $\forall n \in \mathbb{N}$.

Then the sequence $(z_n)_n$, def. by $z_n := \frac{x_n}{\|Lx_n\|}$, is a null sequence, since $\|z_n\| = \frac{\|x_n\|}{\|Lx_n\|} \le \frac{1}{n}$. But $\|Lz_n\| = \frac{\|Lx_n\|}{\|Lx_n\|} = 1$, which contradicts continuity at zero. (10)

Problem 3: Extra Space

Problem 3: Extra Space

b) $A_n \psi := \sum_{k=1}^{n} \langle \varphi_k, \psi \rangle \varphi_k$, with $(\varphi_k)_k$ an ONB.

We compute, for all $\psi \in \mathcal{H}$:

$$\|(A_n - \mathbb{1})\psi\|^2 = \left\| \sum_{k=1}^{n} \langle \varphi_k, \psi \rangle \varphi_k - \psi \right\|^2$$

$$\overset{(\varphi_k)_k \text{ an ONB}}{=} \left\| \sum_{k=1}^{n} \langle \varphi_k, \psi \rangle \varphi_k - \sum_{k=1}^{\infty} \langle \varphi_k, \psi \rangle \varphi_k \right\|^2$$

$$= \left\| \sum_{k=n+1}^{\infty} \langle \varphi_k, \psi \rangle \varphi_k \right\|^2$$

$$= \sum_{k=n+1}^{\infty} |\langle \varphi_k, \psi \rangle|^2$$

$$\xrightarrow{n\to\infty} 0 \quad (\text{since } (\langle \varphi_k, \psi \rangle)_k \in \ell^2).$$

Thus $A_n \xrightarrow{n\to\infty} \mathbb{1}$ strongly. (5)

But: $$\|A_n - \mathbb{1}\| := \sup_{\substack{\phi \in \mathcal{H} \\ \|\phi\| = 1}} \|(A_n - \mathbb{1})\phi\|$$

$$\geq \|(A_n - \mathbb{1})\varphi_{n+1}\|$$

$$= \|\varphi_{n+1}\|$$

$= 1$, so A_n does not converge to $\mathbb{1}$ in operator norm. (5)

Problem 3: Extra Space

Problem 4: Self-adjointness and Unitary Groups [25 points]

Let $\mathcal{L}(\mathcal{H})$ denote the space of bounded operators on a Hilbert space $\mathcal{H}$.

(3) (a) Let $A \in \mathcal{L}(\mathcal{H})$ and let A^* be its Hilbert space adjoint. What is the definition of A self-adjoint? What is the definition of A symmetric? What is the relation between self-adjointness and symmetry for bounded operators?

(2) (b) Define what a unitary operator is.

(2) (c) Let H be a densely defined linear operator with domain $D(H) \subset \mathcal{H}$. Define what it means that H is the generator of a strongly continuous unitary one-parameter group $U(t)$.

(15) (d) Let H with domain $D(H)$ be the generator of $U(t)$. Prove that

- (i) $U(t)D(H) = D(H)$ for all $t \in \mathbb{R}$, (3)
- (ii) $[H, U(t)]\psi := HU(t)\psi - U(t)H\psi = 0$ for all $\psi \in D(H)$, (3)
- (iii) H is symmetric, i.e., $\langle H\psi, \varphi\rangle = \langle \psi, H\varphi\rangle$ for all $\psi, \varphi \in D(H)$, (4)
- (iv) U is uniquely determined by H and H is uniquely determined by U. (5)

(3) (e) Does every bounded linear operator H generate a unitary group? If no, give a counterexample; if yes, define the unitary group which is generated. (But no proofs are necessary here.)

a) A self-adjoint means: $A = A^*$. (1)

A symmetric means: $\langle \psi, A\varphi\rangle = \langle A\psi, \varphi\rangle$ $\forall \psi, \varphi \in \mathcal{H}$. (1)

For bounded operators self-adjoint $\Leftrightarrow$ symmetric. (1)

b) $U \in \mathcal{L}(\mathcal{H})$ is unitary if U is surjective and isometric.

(Alternatively: $U \in \mathcal{L}(\mathcal{H})$ is unitary if $U^{-1} = U^*$.) (2)

c) It means (i) $D(H) = \{\psi \in \mathcal{H} : t \mapsto U(t)\psi$ is differentiable$\}$, (1)

and (ii) For $\psi \in D(H)$, we have $i\frac{d}{dt}U(t)\psi = U(t)H\psi$. (1)

Problem 4: Extra Space

Problem 4: Extra Space

d) (i) $s \mapsto U(s)U(t)\psi = U(s+t)\psi$ is differentiable if and only if

$s \mapsto U(s)\psi = U(-t)U(s+t)\psi$ is differentiable. (3)

(ii) $U(t)H\psi = i\frac{d}{dt}U(t)\psi$

$$= i\frac{d}{dt}U(t+s)\psi\Big|_{s=0}$$

$$= i\frac{d}{ds}U(s)U(t)\psi\Big|_{s=0}$$

$$= HU(t)\psi. \qquad (3)$$

(iii) For $\varphi, \psi \in D(H)$, we can compute:

$$0 = \frac{d}{dt}\langle \varphi, \psi\rangle \underset{U(t)\text{ unitary}}{=} \frac{d}{dt}\langle U(t)\varphi, U(t)\psi\rangle$$

$$= \langle -iU(t)H\varphi, U(t)\psi\rangle + \langle U(t)\varphi, (-i)U(t)H\psi\rangle$$

$$= i\langle H\varphi, \psi\rangle - i\langle \varphi, H\psi\rangle \qquad (4)$$

(iv) Suppose $U(t)$ and $\tilde{U}(t)$ are generated by H. Then, for $\psi \in D(H)$,

$$\frac{d}{dt}\|(U(t)-\tilde{U}(t))\psi\|^2 = \frac{d}{dt}\left[2\mathrm{Re}\left(\|\psi\|^2 - \langle U(t)\psi, \tilde{U}(t)\psi\rangle\right)\right]$$

$$= -2\mathrm{Re}\left(\langle -iHU(t)\psi, \tilde{U}(t)\psi\rangle + \langle U(t)\psi, (-i)H\tilde{U}(t)\psi\rangle\right)$$

$$= 0 \text{ by symmetry of } H.$$

Now $(U(0)-\tilde{U}(0))\psi = 0$, so $\tilde{U}|_{D(H)} = U|_{D(H)}$, and $U = \tilde{U}$ on $\mathcal{H}$ because $D(H)$ is dense in $\mathcal{H}$! (4) H is uniquely determined by U by definition. (1)

Problem 4: Extra Space

e) Note: Every <u>self-adjoint</u> $H \in \mathcal{S}(\mathcal{H})$ generates the unitary group $e^{-iHt} := \sum_{n=0}^{\infty} \frac{(-iHt)^n}{n!}$.

But, e.g., the operator of multiplication with i does not generate a group $U(t)$, e.g., since it is not symmetric. (3)

Bonus Problem: Second Quantization [~~20~~ 25 points]

Let $\mathcal{F}$ be the (bosonic) Fock space as defined in class, and let $f \in L^2(\mathbb{R}^d)$. Recall that we defined the annihilation operator $a(f) : \mathcal{F} \to \mathcal{F}$ by

$$\left(a(f)\chi\right)^{(k)}(x_1, \ldots, x_k) = \sqrt{k+1} \int \mathrm{d}x\, \overline{f(x)} \chi^{(k+1)}(x_1, \ldots, x_k, x)$$

for any $\chi \in \mathcal{F}$. (Here, $\chi^{(k)}$ denotes the k-particle sector of $\chi \in \mathcal{F}$, and χ is symmetric under exchange of all variables.)

(10) (a) Compute the action of the adjoint $a^*(f)$ on any $\chi \in \mathcal{F}$ in the k-particle sector, i.e., compute $\left(a^*(f)\chi\right)^{(k)}(x_1, \ldots, x_k)$.

(8) (b) Let $(\varphi_n)_{n\in\mathbb{N}}$ be an orthonormal basis, and define

$$\mathcal{N} := \sum_{n\in\mathbb{N}} a^*(\varphi_n) a(\varphi_n).$$

Prove that $\left(\mathcal{N}\chi\right)^{(k)} = k\chi^{(k)}$ for any $\chi \in \mathcal{F}$.

(7) (c) Let $\|\chi\|_{\mathcal{F}} = 1 = \|f\|_{L^2(\mathbb{R}^d)}$. Prove that

$$\|a(f)\chi\|^2_{\mathcal{F}} \leq \langle \chi, \mathcal{N}\chi \rangle$$

and

$$\|a^*(f)\chi\|^2_{\mathcal{F}} \leq \langle \chi, (\mathcal{N}+1)\chi \rangle.$$

a) For all $\psi, \chi \in \mathcal{F}$ we have

$$\langle a(f)\psi, \chi \rangle = \langle \psi, a^*(f)\chi \rangle$$

$$= \sum_{k=0}^{\infty} \langle (a(f)\psi)^{(k)}, \chi^{(k)} \rangle$$

$$= \sum_{k=0}^{\infty} \int dx_1 \ldots dx_k \sqrt{k+1} \overline{\int dx\, \overline{f(x)}\, \psi^{(k+1)}(x_1, \ldots, x_k, x)}\, \chi^{(k)}(x_1, \ldots, x_k)$$

$$= \sum_{k=0}^{\infty} \int dx_1 \ldots dx_k dx_{k+1} \sqrt{k+1}\, \overline{\psi^{(k+1)}(x_1, \ldots, x_k, x_{k+1})}\, f(x_{k+1}) \chi^{(k)}(x_1, \ldots, x_k)$$

$$= \sum_{k=0}^{\infty} \int dx_1 \ldots dx_{k+1} \sqrt{k+1}\, \overline{\psi^{(k+1)}(x_1, \ldots, x_{k+1})} \sum_{m=1}^{k+1} f(x_m) \chi^{(k)}(x_1, \ldots, x_{m-1}, x_{m+1}, \ldots x_{k+1}) \cdot \frac{1}{k+1}$$

$k+1 \to j$
$$= \sum_{j=1}^{\infty} \int dx_1 \ldots dx_j\, \overline{\psi^{(j)}(x_1, \ldots, x_j)}\, \frac{1}{\sqrt{j}} \sum_{m=1}^{j} f(x_m) \chi^{(j-1)}(x_1, \ldots, x_{m-1}, x_{m+1}, \ldots, x_j)$$

i.e., $\left(a^*(f)\chi\right)^{(k)}(x_1, \ldots, x_k) = \frac{1}{\sqrt{k}} \sum_{m=1}^{k} f(x_m) \chi^{(k-1)}(x_1, \ldots, x_{m-1}, x_{m+1}, \ldots, x_k)$. (10)

Bonus Problem: Extra Space

Bonus Problem: Extra Space

b) $(\mathcal{N}\chi)^{(k)}(x_1,\dots,x_k) := \sum_n \left(a^*(\varphi_n)a(\varphi_n)\chi\right)^{(k)}(x_1,\dots,x_k)$

$$= \sum_n \sum_{j=1}^{k} \varphi_n(x_j) \int dx\, \overline{\varphi_n(x)}\, \chi^{(k)}(x_1,\dots,x_{j-1},x_{j+1},\dots,x_k,x)$$

def. of ONB

$$= \sum_{j=1}^{k} \chi^{(k)}(x_1,\dots,x_{j-1},x_{j+1},\dots,x_k,x_j)$$

$\chi^{(k)}$ symmetric

$$= k\, \chi^{(k)}(x_1,\dots,x_k) \quad (8)$$

c) $\|a(f)\chi\|^2 = \langle a(f)\chi, a(f)\chi\rangle = \langle \chi, a^*(f)a(f)\chi\rangle \leq \langle \chi, \mathcal{N}\chi\rangle$

(e.g., by choosing an ONB with $\varphi_1 = f$). (3)

$\|a^*(f)\chi\|^2 = \langle a^*(f)\chi, a^*(f)\chi\rangle = \langle \chi, a(f)a^*(f)\chi\rangle$

$[a(f), a^*(f)] = 1$ for $\|f\| = 1$

$$= \langle \chi, (1 + a^*(f)a(f))\chi\rangle$$

$$\leq \langle \chi, (1+\mathcal{N})\chi\rangle. \quad (4)$$

Bonus Problem: Extra Space