Constructor University — September 8, 2023
Fall 2023

Foundations of Mathematical Physics

Homework 1

Due on Sep. 20, 2023, before the tutorial.

Problem 1 [6 points]: Bosons and Fermions
We consider two non-interacting particles in one dimension confined to the box $[-\frac{1}{2}, \frac{1}{2}]$, with periodic boundary conditions.

(a) Consider just one of the particles, and find the ground state $\varphi_0(x)$ and first excited state $\varphi_1(x)$. (The time-independent Schrödinger equation is $E\varphi = -\frac{1}{2}\Delta\varphi$.)

(b) For fermions, the 2-particle ground state is

$$\psi_{\text{fermion}}(x_1, x_2) = \frac{1}{\sqrt{2}}\Big(\varphi_0(x_1)\varphi_1(x_2) - \varphi_1(x_1)\varphi_0(x_2)\Big).$$

Consider also the bosonic state with the same energy (which here is the first 2-particle excited state)

$$\psi_{\text{boson}}(x_1, x_2) = \frac{1}{\sqrt{2}}\Big(\varphi_0(x_1)\varphi_1(x_2) + \varphi_1(x_1)\varphi_0(x_2)\Big).$$

For each of these two states, what is the probability that both particles are on the same side of the box, i.e., that *both* particles are in $[-\frac{1}{2}, 0]$ or in $[0, \frac{1}{2}]$?

(c) Compute the probability above also for the bosonic 2-particle ground state.

Problem 2 [8 points]: Schrödinger's Equation

(a) [**Finite Square Well**]. Let $0 < a < b$. Solve the one-dimensional Schrödinger equation for the potential

$$V(x) := \begin{cases} b & , x \leq 0 \\ -a & , 0 \leq x \leq L \\ b & , L \leq x. \end{cases}$$

(b) [**Particle in a Box**] Solve the one-dimensional Schrödinger equation for the potential

$$W(x) := \begin{cases} \infty & , L \leq x \leq 0 \\ -a & , 0 \leq x \leq L. \end{cases}$$

(c) Using a) and b) deduce that in the limit $b \to \infty$, the wave function for the finite square well converges pointwise to the wave function for the particle in a box.

Hint: Since the potentials are piece-wise defined, solve the differential equation piece-wise and remember that the wave function needs to be normalized and assume that it is continuous.

Problem 3 [6 points]: Metric Spaces and Limits

(a) Let (X, d) be a metric space. Define the notion of limit on a general metric space. What does it mean for a set $A \subset X$ to be dense in X with respect to the metric d?

(b) Let $C([a, b])$ be the set of all real-valued continuous functions defined on $[a, b]$ and $d : C([a, b]) \times C([a, b]) \to [0, \infty)$ be the supremum metric

$$d(f, g) := \max_{x \in [a,b]} |f(x) - g(x)|.$$

Show that $(C([a, b]), d)$ is a metric space. Also explain why, in this case, it makes sense to replace the supremum by maximum in the definition of supremum metric.

(c) Define the notion of pointwise continuity. Prove that continuity with respect to the supremum metric implies pointwise continuity but not the other way around.

(d) Define another metric on $C([a, b])$ and deduce whether convergence with respect to the supremum metric implies convergence with respect to the defined metric.