

Foundations of Mathematical Physics

Homework 3

Due on Oct. 4, 2023, before the tutorial.

Problem 1 [6 points]: Bijectivity of the Fourier Transform

Finish the proof that was sketched in class, that the Fourier transform $\mathcal{F} : \mathcal{S} \rightarrow \mathcal{S}$ is a continuous bijection with continuous inverse \mathcal{F}^{-1} (with \mathcal{F}^{-1} as defined in class).

Problem 2 [2 points]: Plancherel on \mathcal{S}

Prove that for $f, g \in \mathcal{S}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \widehat{f}(x)g(x)dx = \int_{\mathbb{R}^d} f(x)\widehat{g}(x)dx,$$

and in particular,

$$\int_{\mathbb{R}^d} |\widehat{f}(k)|^2 dk = \int_{\mathbb{R}^d} |f(x)|^2 dx.$$

Problem 3 [6 points]: Dilations

Let $p \in [1, \infty)$, $\sigma > 0$ and $D_\sigma^p : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$, $f(x) \mapsto (D_\sigma^p f)(x) = \sigma^{-d/p} f(x/\sigma)$ the L^p dilation with σ .

- (a) Show that $D_\sigma^p : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$ is continuous and $\|D_\sigma^p f\|_{L^p(\mathbb{R}^d)} = \|f\|_{L^p(\mathbb{R}^d)}$.
- (b) Compute $\mathcal{F}D_\sigma^2 f$ and interpret the result.

Problem 4 [6 points]: Convolution

Let $f \in C(\mathbb{R}^d)$ be bounded and let $\varphi \in \mathcal{S}(\mathbb{R}^d)$ with $\varphi \geq 0$ and $\|\varphi\|_{L^1(\mathbb{R}^d)} = 1$. Define $f_\sigma := f * (D_\sigma^1 \varphi)$, where the D_σ^1 are the dilations from Problem 3.

- (a) Show that $f_\sigma \in C^\infty(\mathbb{R}^d)$.
- (b) Show that $f_\sigma(x) \rightarrow f(x)$ as $\sigma \rightarrow 0$ for all $x \in \mathbb{R}^d$.
- (c) Show that the convergence in b) is uniform on each compact interval. You may additionally assume $\varphi \in C_c^\infty(\mathbb{R}^d)$.