Constructor University
Fall 2023

November 15, 2023

Foundations of Mathematical Physics

Homework 10

Due on Nov. 23, 2023, before the tutorial.

Problem 1 [6 points]: Properties of the Hilbert space adjoint

Prove the following properties of the Hilbert space adjoint that were stated in class. Let A, B be bounded linear operators on a Hilbert space $\mathcal{H}$, and $\lambda \in \mathbb{C}$. Then

(a) $(A+B)^* = A^* + B^*$ and $(\lambda A)^* = \overline{\lambda} A^*$,

(b) $(AB)^* = B^* A^*$,

(c) $\|A^*\| = \|A\|$,

(d) $A^{**} = A$,

(e) $\|AA^*\| = \|A^*A\| = \|A\|^2$,

(f) $\ker A = (\operatorname{im} A^*)^\perp$ and $\ker A^* = (\operatorname{im} A)^\perp$.

Problem 2 [8 points]: Unitary groups with bounded generators

Let $\mathcal{H}$ be a Hilbert space and let $H \in \mathcal{L}(\mathcal{H})$ be symmetric. Prove that

$$e^{-iHt} := \sum_{n=0}^{\infty} \frac{(-iHt)^n}{n!}$$

defines a unitary group generated by $(H, \mathcal{H})$ which is also uniformly differentiable in t. *Hint: Proceed step by step: Show that the series is well-defined, that the group property holds, unitarity, uniform differentiability, and finally that the Schrödinger equation holds.*

Problem 3 [6 points]: Neumann series

Let X be a Banach space and $T \in \mathcal{L}(X)$ with $\|T\| < 1$. Prove that $1 - T$ is invertible with inverse

$$(1-T)^{-1} = \sum_{n=0}^{\infty} T^n.$$

Conclude that $T \in \mathcal{L}(X)$ is invertible if $\|1 - T\| < 1$. *Hint: Geometric series.*