Constructor University
Fall 2023

November 29, 2023

Foundations of Mathematical Physics

Homework 12

Due on Dec. 7, 2023, before the tutorial.

Problem 1 [6 points]: Fermionic reduced density matrix

Let $\Psi_N \in L^2(\mathbb{R}^{3N})$ be antisymmetric in $x_1, \dots, x_N$ (fermionic wave function), and $\|\Psi_N\| = 1$. Prove that the one-particle reduced density matrix γ_{Ψ_N} satisfies $\|\gamma_{\Psi_N}\|_{\mathcal{L}} \leq N^{-1}$.
Hint: You can use that

$$\|A\|_{\mathcal{L}} = \sup_{\varphi \in L^2, \|\varphi\|=1} \langle \varphi, A\varphi \rangle$$

for all non-negative $A \in \mathcal{L}(L^2)$ (where A is called non-negative if $\langle \varphi, A\varphi \rangle \geq 0$ for all $\varphi \in L^2$.) Then show that $\sum_{i=1}^N p_i^\varphi$ is a projector on antisymmetric $L^2(\mathbb{R}^{3N})$ functions.

Problem 2 [7 points]: Energy conservation

Let $\varphi(t) \in L^2(\mathbb{R}^3)$ be solution to the Hartree equation

$$i\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) = -\Delta\varphi(t) + (v * |\varphi(t)|^2)\varphi(t)$$

for bounded even real-valued v. We assume a solution exists and $\varphi(t) \in H^2(\mathbb{R}^3)$. We define the energy

$$E(\varphi(t)) := \|\nabla\varphi(t)\|^2 + \frac{1}{2}\int (v * |\varphi(t)|^2)(x)\,|\varphi(t,x)|^2\,\mathrm{d}x.$$

Prove energy conservation, i.e., that $E(\varphi(t)) = E(\varphi(0))$.

Problem 3 [7 points]: Term (II)

For any symmetric $\Psi_N \in L^2(\mathbb{R}^{3N})$ and any $\varphi \in L^2(\mathbb{R}^3)$ with $\|\Psi_N\| = 1 = \|\varphi\|$, and $v \in L^\infty$, prove that

$$\Big|\langle \Psi_N, p_1^\varphi p_2^\varphi v_{12} q_1^\varphi q_2^\varphi \Psi_N \rangle\Big| \leq 3\|v\|_\infty\Big(\alpha(\Psi_N,\varphi) + N^{-1}\Big).$$

Hint: Symmetrize the term, i.e., write

$$\langle \Psi_N, p_1^\varphi p_2^\varphi v_{12} q_1^\varphi q_2^\varphi \Psi_N \rangle = \frac{1}{N-1}\langle \Psi_N, \sum_{i=2}^N p_1^\varphi p_i^\varphi v_{1i} q_i^\varphi q_1^\varphi \Psi_N \rangle,$$

and then use Cauchy-Schwarz in a clever way.