2. The Free Schrödinger Equation

Central topic of this class:

For which $\psi(t=0)$ and V does Schrödinger equation have global solutions, and in which sense?

General idea: regard Schrödinger equation as an ODE $i\frac{d}{dt}\psi(t) = H\psi(t)$ for

$$\psi : \mathbb{R} \to \mathcal{H} = \text{some function space, or better, Hilbert space}$$

Difficulties: • $\mathcal{H}$ infinite dimensional
• H unbounded

Since we want $\int |\psi|^2 = 1$, we need $\psi \in L^2((\mathbb{R}^d)^N)$.

More generally, let us consider the following function spaces:

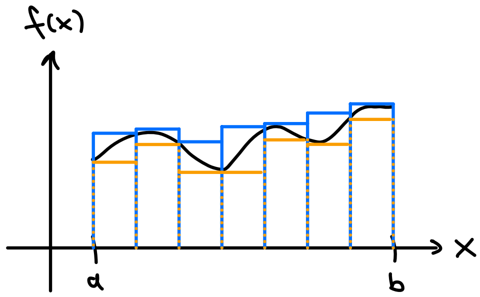
$$\boxed{L^p(\mathbb{R}^d) := \left\{ f : \mathbb{R}^d \to \mathbb{C} : \underbrace{\|f\|_p}_{p\text{-norm}} = \left(\int |f|^p\right)^{\frac{1}{p}} < \infty \right\}}, \quad 1 \le p < \infty$$

Here, all integrals refer to the $\boxed{\text{Lebesgue integral}}$.

Today: Quick introduction to the Lebesgue integral (here just for $d=1$)

Recall: $f:[a,b]\to\mathbb{R}$ bounded is Riemann integrable if upper and lower Riemann integrals coincide

$$\text{upper: } \inf_{\substack{\text{partitions}\\ \text{of } [a,b]}} \sum_{i=1}^{m} \underbrace{M_i}_{=\sup_{x\in[x_{i-1},x_i]} f(x)} \Delta x_i \qquad \text{lower: } \sup_{\substack{\text{partitions}\\ \text{of } [a,b]}} \sum_{i=1}^{m} \underbrace{m_i}_{=\inf_{x\in[x_{i-1},x_i]} f(x)} \Delta x_i$$



Examples: • continuous functions are Riemann integrable

• $\mathbb{1}_{\mathbb{Q}}|_{[0,1]}$ is not Riemann integrable

↳ note: for $S\subset\mathbb{R}^d$, we define the indicator function $\mathbb{1}_S(x) := \begin{cases} 1 & \text{for } x\in S \\ 0 & \text{for } x\notin S\end{cases}$

Note: Improper Riemann integrals ($a=-\infty$ or $b=\infty$ or f is not bounded) might exist as limits of Riemann integrals

Recall the following result on interchanging limits and integration:

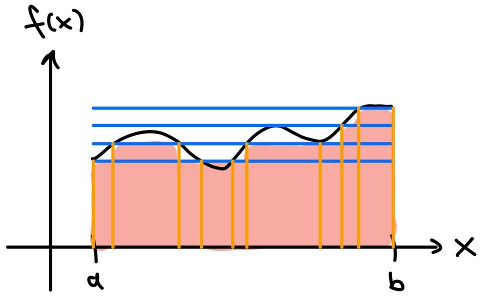
If $(f_n)_n \xrightarrow{n\to\infty} f$ uniformly, then $\lim\limits_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \underbrace{\lim_{n\to\infty} f_n(x)}_{f(x)}\,dx$.

But this might fail for improper integrals, e.g., $\lim\limits_{n\to\infty} \int_{-\infty}^{\infty} \underbrace{n^{-1}\mathbb{1}_{[0,n]}(x)}_{\text{converges uniformly to zero}}\,dx = 1 \neq 0$.

The Lebesgue integral addresses the difficulties with exchanging limits and integration.

Let us first go through the idea of the construction.

Main idea: We partition y-axis instead of x-axis



a collection of subsets of Ω

$\mu: \Sigma \to \mathbb{R}_+$ satisfying reasonable axioms

e.g., $\Omega = \mathbb{R}$

Steps in constructing the Lebesgue integral:

- Define $\underbrace{\text{"size"}}_{=\text{measure}}$ of a subset $S \subset \mathbb{R}$; this leads to measure spaces (Ω, Σ, μ); e.g., $\mu([a,b]) = b-a$.

- Approximate f by "simple functions" $\sum_k a_k \mathbb{1}_{S_k}$ (S_k measurable)
  ↳ then $\int \sum_k a_k \mathbb{1}_{S_k} \, d\mu = \sum_k a_k \mu(S_k)$

- Then $\int f \, d\mu := \sup \{ \int s \, d\mu : 0 \leq s \leq f, s \text{ simple} \}$ for $f \geq 0$

- In general: $\int f \, d\mu := \underbrace{\int f^+ d\mu}_{\text{positive part of } f} - \underbrace{\int f^- d\mu}_{\text{negative part of } f}$ if one of the integrals is finite

Note:
- $f: [a,b] \to \mathbb{R}$ (bounded) Riemann integrable $\Rightarrow f$ Lebesgue integrable
- $\int \mathbb{1}_{\mathbb{Q}}|_{[0,1]} \, d\mu = 0$ (bc. $\mu(\mathbb{Q} \cap [0,1]) = 0$)
- But there are improper well-defined Riemann integrals that do not exist as Lebesgue integrals

Important theorems about Lebesgue integration:

$\boxed{\text{Monotone Convergence:}}$ If $(f_n)_n$ with $f_n \geq 0$ and f_n measurable is such that $f_n(x) \leq f_{n+1}(x)$ $\forall n \in \mathbb{N}$ $\forall x \in \mathbb{R}$, then $\lim_{n\to\infty} \int f_n \, d\mu = \int \underbrace{\lim_{n\to\infty} f_n}_{\text{pointwise limit}} \, d\mu$

Dominated Convergence: If $(f_n)_n$ with f_n measurable $\forall n$ and $\lim_{n\to\infty} f_n(x) = f(x)$ is such that $|f_n(x)| \le g(x)$ $\forall n \in \mathbb{N}$ $\forall x \in \mathbb{R}$ for some measurable g with $\int |g| < \infty$, then $\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu$.

Note: Dominated Convergence still holds if $f_n(x) \to f(x)$ and $|f_n(x)| \le g(x)$ holds $\forall n \in \mathbb{N}$ for $\underbrace{\text{almost all } x}_{\text{"almost everywhere"}}$, i.e., for all x except those in $\underbrace{\text{some set of measure zero}}_{\text{e.g., finitely many points}}$.

note: we often abbreviate:
- almost all x = a.a. x
- almost everywhere = a.e.

Fubini: If f is measurable with $\iint_{\mathbb{R}\times\mathbb{R}} |f(x,y)| \, dx\,dy < \infty$, then

$$\iint_{\mathbb{R}\times\mathbb{R}} f(x,y) \, dx\,dy = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y)\,dy \right) dx = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y)\,dx \right) dy.$$

From now on, all integrals are meant in the Lebesgue sense, and we use the usual notations $\int f \, d\mu \equiv \int f(x)\,dx \equiv \int dx \, f(x)$.