Recall: we defined $L^p(\mathbb{R}^d) := \left\{ f: \mathbb{R}^d \to \mathbb{C} : \|f\|_p := \Big(\underbrace{\int |f(x)|^p dx}_{\text{Lebesgue integral}} \Big)^{\frac{1}{p}} < \infty \right\}$

Remarks:

- $L^\infty(\mathbb{R}^d) := \left\{ f: \mathbb{R}^d \to \mathbb{C} : \|f\|_\infty := \overbrace{\inf\{C \geq 0 : |f(x)| \leq C \text{ for almost all } x\}}^{=: \text{ess sup } f \text{ (essential supremum)}} < \infty \right\}$

  Note: one can show that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$ $\left(\forall f \in L^\infty \cap L^q \text{ for some } q\right)$

- For all $1 \leq p \leq \infty$, $L^p(\mathbb{R}^d)$ are Banach spaces (i.e., complete normed vector spaces) if one identifies functions that agree almost everywhere (always assumed)
  $\Rightarrow$ really, $L^p(\mathbb{R}^d)$ are vector spaces of equivalence classes of functions

- Only $L^2(\mathbb{R}^d)$ is a $\underbrace{\text{Hilbert space}}_{}$ with scalar product $\langle f, g \rangle = \int \overline{f} g$
  = Banach space with norm given by scalar product, i.e., $\|f\|^2 = \langle f, f \rangle$

2.1 Fourier Transform on Schwartz Space

Notes:
- From now on we use natural units, i.e., $\hbar = m = 1$.
- For $\psi: \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$, partial derivatives are defined in the usual way:
  $\partial_x \psi(t,x) := \partial_x \operatorname{Re}\psi(t,x) + i\, \partial_x \operatorname{Im}\psi(t,x)$

Free one-particle SE: $V = 0$, i.e., $i\partial_t \psi(t,x) = -\frac{1}{2}\Delta_x \psi(t,x)$, $\psi: \mathbb{R}_t \times \mathbb{R}^d_x \to \mathbb{C}$

Recall: solutions to the stationary SE $-\frac{1}{2}\Delta_x \phi(x) = E\,\phi(x)$
give us solutions $\psi(t,x) = e^{-iEt}\phi(x)$

Formally, the "eigenfunctions" of $-\frac{1}{2}\Delta_x$ are $\boxed{\text{plane waves}}$

$\boxed{\phi_k(x) = e^{ik\cdot x}} = e^{i\sum_{j=1}^{d} k_j x_j}$, for any $k \in \mathbb{R}^d$ $\left(\text{since } -\frac{1}{2}\Delta_x \phi_k(x) = \frac{1}{2}k^2 \phi_k(x)\right)$

$\Rightarrow$ this gives solutions $\psi_k(t,x) = e^{-i\frac{k^2}{2}t} e^{ikx}$ of the free SE

But $|\psi_k(t,x)|^2 = 1$, so on $\mathbb{R}^d$ $\int_{\mathbb{R}^d} |\psi_k(t,x)|^2 dx = \infty$, but we want $\int_{\mathbb{R}^d} |\psi|^2 = 1$.

By linearity, we find that formally $\psi(t,x) = \int f(k)\psi_k(t,x)\,dk = \int f(k) e^{-i\frac{k^2}{2}t} e^{ikx} dk$

is also a solution, and $f: \mathbb{R}^d \to \mathbb{C}$ is determined by the initial condition:

$\psi(0,x) = \int f(k) e^{ikx} dk$

Conclusion: we need to study the Fourier transform on $\mathbb{R}^d$.

First step: we define the Fourier transform on L^1.

<u>Def. 2.1:</u> Let $f, g \in L^1(\mathbb{R}^d)$, then we define the

- Fourier transform of f as $\boxed{\hat{f}(k) = (\mathcal{F}f)(k) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(x) e^{-ikx} dx,}$
- inverse Fourier transform of g as $\boxed{\check{g}(x) = (\mathcal{F}^{-1}g)(x) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} g(k) e^{ikx} dk.}$

(Note: We do not know yet in what sense $\mathcal{F}^{-1}$ is the inverse of $\mathcal{F}$.)

Next: we want to know about regularity of $\hat{f}$ (i.e., continuity, differentiability)

$\to$ need to take derivative of integral with parameter

Lemma 2.2: Integrals with Parameter

Let $I(\gamma) = \int_{\mathbb{R}^d} f(x,\gamma)\,dx$, with $f: \mathbb{R}^d \times \Gamma \to \mathbb{C}$, where $\Gamma \subset \mathbb{R}$ an open interval, and let $f(x,\gamma) \in L^1(\mathbb{R}^d)$ for all fixed $\gamma \in \Gamma$.

a) If $\gamma \mapsto f(x,\gamma)$ is continuous for $\overbrace{\text{almost all}}^{:=\text{a.a.}}$ $x \in \mathbb{R}^d$

and if $\exists\, g \in L^1(\mathbb{R}^d)$ with $\sup_{\gamma\in\Gamma} |f(x,\gamma)| \le g(x)$ for a.a. $x \in \mathbb{R}^d$,

then $I(\gamma)$ is continuous.

b) If $\gamma \mapsto f(x,\gamma)$ is continuously differentiable for a.a. $x \in \mathbb{R}^d$

and if $\exists\, g \in L^1(\mathbb{R}^d)$ with $\sup_{\gamma\in\Gamma} |\partial_\gamma f(x,\gamma)| \le g(x)$ for a.a. $x \in \mathbb{R}^d$,

then $I(\gamma)$ is continuously differentiable and

$$\frac{dI(\gamma)}{d\gamma} = \frac{d}{d\gamma}\int_{\mathbb{R}^d} f(x,\gamma)\,dx = \int_{\mathbb{R}^d} \partial_\gamma f(x,\gamma)\,dx.$$

Proof: HW. Use dominated convergence.

(Note: Lemmas like this one are one of the main advantages of Lebesgue over Riemann integral.)