We need to introduce more notation:

- a $\boxed{\text{multi-index } \alpha \in \mathbb{N}_0^d}$ is a typel $(\alpha_1, \dots, \alpha_d)$, $\alpha_j \in \mathbb{N}_0$.
  We denote $|\alpha| := \sum_{j=1}^d \alpha_j$, and for $x \in \mathbb{R}^d$, $x^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}$, $\partial_x^\alpha := \dfrac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}}$.

- $\boxed{C^p(\mathbb{R}^d)} := \Big\{ f : \underbrace{\partial_x^\alpha f \text{ continuous } \forall \text{ multi-indices } \alpha \text{ with } |\alpha| \le p}_{f \text{ p times continuously differentiable}} \Big\}$

- $\boxed{C^\infty(\mathbb{R}^d)} = \bigcap_{p \in \mathbb{N}} C^p(\mathbb{R}^d)$ = smooth functions (∞ often continuously differentiable)
- $C^0(\mathbb{R}^d) = \boxed{C(\mathbb{R}^d)}$ = continuous functions
- $\boxed{\underbrace{C_\infty(\mathbb{R}^d)}_{\text{sometimes called } C_0(\mathbb{R}^d)}} := \Big\{ f \in C(\mathbb{R}^d) : \underbrace{\lim_{|x| \to \infty} f(x) = 0}_{\text{more exact: } \lim_{R \to \infty} \sup_{|x| > R} |f(x)| = 0} \Big\}$

- $\boxed{C_c^p(\mathbb{R}^d)} := C^p(\mathbb{R}^d) \cap \Big\{ f : \underbrace{\text{supp} f}_{\text{support of } f} \underbrace{\text{compact}}_{\text{in } \mathbb{R}^d, \text{ compact} \Leftrightarrow \text{closed and bounded}} \Big\}$ = functions with compact support

Where does the Fourier transform on L^1 map to? We know the following:

> **Lemma 2.3: Riemann-Lebesgue**
>
> $$f \in L^1(\mathbb{R}^d) \Rightarrow \hat{f} \in C_\infty(\mathbb{R}^d).$$

Proof: • $f \in L^1(\mathbb{R}^d)$, recall $\hat{f}(k) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \underbrace{f(x) e^{-ikx}}\, dx$

↳ cont. in k for a.a. x

↳ $\sup_k |f(x) e^{-ikx}| = |f(x)| \in L^1(\mathbb{R}^d)$

$\Rightarrow \hat{f}$ continuous with Lemma 2.2.

- Decay at ∞ follows later from a more general result. $\square$

Want: $\mathcal{F}$ maps from a fct. space X to itself (and $\mathcal{F}^{-1}: X \to X$ is the inverse).
So for now we go away from L^1 and instead consider the following class of very nice functions.

Definition 2.5: Schwartz space

We call the $\mathbb{C}$-vector space

$$S(\mathbb{R}^d) := \{ f \in C^\infty(\mathbb{R}^d) : \|f\|_{\alpha,\beta} < \infty \ \forall \text{ multi-indices } \alpha, \beta \in \mathbb{N}_0^d \}$$ $\boxed{\text{Schwartz space}}$

(space of smooth rapidly decaying functions). Here,

$$\|f\|_{\alpha,\beta} := \|x^\alpha \partial_x^\beta f(x)\|_\infty = \sup_{x \in \mathbb{R}^d} |x^\alpha \partial_x^\beta f(x)|.$$

Note:
- for $f \in S(\mathbb{R}^d)$, f and all partial derivatives decay faster than any polynomial
- e.g., $e^{-x^2} \in S(\mathbb{R}^d)$, $C_c^\infty(\mathbb{R}^d) \subset S(\mathbb{R}^d)$

Definition: On a vector space V, a map $\|\cdot\|: V \to \mathbb{R}_{\geq 0}$ is called semi-norm if
- $\|\lambda f\| = |\lambda| \cdot \|f\|$ (absolute homogeneity)
- $\|f+g\| \leq \|f\| + \|g\|$ (triangle inequality)

Note:
- for a norm, we require additionally that $\|f\| = 0 \Rightarrow f = 0$
- $\|f\|_{\alpha,\beta}$ are semi-norms (for $\beta = 0$, $\|f\|_{\alpha,0}$ is also a norm)
  - ↳ e.g., $d=1$, $\|x\|_{0,2} = \|\partial_x^2 x\|_\infty = 0$ (but $f(x) = x \not\equiv 0$)

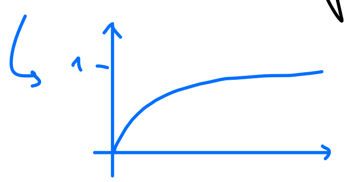
Next: Since we have only a family of semi-norms on $\mathcal{S}$, it is not a Banach space; but we can construct a complete metric space (in this context called a Fréchet space) in the following way.

Lemma 2.8:

$$d_{\mathcal{S}}(f,g) := \sum_{n=0}^{\infty} 2^{-n} \sup_{|\alpha|+|\beta|=n} \left(\frac{\|f-g\|_{\alpha,\beta}}{1+\|f-g\|_{\alpha,\beta}} \right)$$ is a metric on $\mathcal{S}$.

Note: the choice of $\frac{\|\cdots\|_{\alpha,\beta}}{1+\|\cdots\|_{\alpha,\beta}}$ is a convention; we could choose other functions that lead to the triangle inequality and go to zero for $\|\cdots\|_{\alpha,\beta}$ going to zero.

Proof: First, note that $\frac{x}{1+x}$ maps $\mathbb{R}_{\geq 0}$ to $[0,1]$ and is monotonically increasing.



$$\Rightarrow d_{\mathcal{S}}(f,g) \leq \sum_{n=0}^{\infty} 2^{-n} = \frac{1}{1-\frac{1}{2}} = 2$$

We now check the properties of a metric:

- $d_{\mathcal{S}}(f,g) \geq 0$ clear
- $d_{\mathcal{S}}(f,g) = d_{\mathcal{S}}(g,f)$ clear
- $d_{\mathcal{S}}(f,g) = 0 \Leftrightarrow f = g$?

  ↳ "$\Leftarrow$" clear

  ↳ "$\Rightarrow$" let $d_{\mathcal{S}}(f,g) = 0$;

  then in particular $\|f-g\|_{0,0} = \|f-g\|_{\infty} := \sup_{x\in\mathbb{R}^d} |f(x)-g(x)| = 0 \Rightarrow f = g$

- $d_{\mathcal{S}}(f,g) \leq d_{\mathcal{S}}(f,h) + d_{\mathcal{S}}(h,g)$?

  ↳ we have $\|f-g\|_{\alpha,\beta} = \|f-h+h-g\|_{\alpha,\beta} \leq \underbrace{\|f-h\|_{\alpha,\beta}}_{:=x} + \underbrace{\|h-g\|_{\alpha,\beta}}_{:=y}$

$\hookrightarrow$ then $\dfrac{\|f-g\|_{\alpha,\beta}}{1+\|f-g\|_{\alpha,\beta}} \overset{\text{monotone increasing}}{\leq} \dfrac{x+y}{1+x+y} = \dfrac{x}{1+x+y} + \dfrac{y}{1+x+y} \leq \dfrac{x}{1+x} + \dfrac{y}{1+y}$ ✓ $\square$

<u>Corollary:</u> Convergence in $\mathcal{S}$

$$f_n \xrightarrow{n\to\infty} f \text{ in } \mathcal{S} :\Longleftrightarrow d_{\mathcal{S}}(f,f_n) \xrightarrow{n\to\infty} 0$$

$$\Longleftrightarrow \|f-f_n\|_{\alpha,\beta} \xrightarrow{n\to\infty} 0 \quad \forall \alpha,\beta \in \mathbb{N}_0^d .$$