We continue our study of (S, d_S).

An important property is:

Every Cauchy sequence converges.

$\boxed{\text{Lemma 2.9: The metric space } (S, d_S) \text{ is complete.}}$

Recall: • $(f_m)_m$ is a Cauchy sequence means: $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ s.t. $d(f_m, f_n) < \varepsilon\ \forall m, n > N$

• Clearly every convergent sequence is also a Cauchy sequence since
$d(f_m, f_n) \leq d(f_m, f) + d(f_n, f)$ (if RHS $\to 0$ then also LHS $\to 0$)

• In the definition of a Cauchy sequence we only use the f_m (not a possible limit f); this is technically nice and often easier to work with. If completeness holds (i.e., $(f_m)_m$ Cauchy $\Leftrightarrow$ $(f_m)_m$ converges), we just have to check the Cauchy property and then know that a limit exists.

Proof: Let $(f_m)_m$ be a Cauchy sequence in S.

Idea: We first construct a candidate f for the limit, and then show that it is indeed the limit in S.

Note: $(f_m)_m$ Cauchy in S $\Rightarrow$ (f_m) is also Cauchy w.r.t. all $\|\cdot\|_{\alpha,\beta}$;

put differently: $f_m^{(\alpha,\beta)}(x) := x^\alpha \partial_x^\beta f_m(x)$ is Cauchy w.r.t. $\|\cdot\|_\infty$.

We use the result from Analysis that $C_b(\mathbb{R}^d) := \{ f \in C(\mathbb{R}^d) : f \text{ bounded} \}$ is complete w.r.t. $\|\cdot\|_\infty$. Thus $f_m^{(\alpha,\beta)} \xrightarrow{m\to\infty} f^{(\alpha,\beta)}$ uniformly. (See, e.g., Rudin: Principles of Mathematical Analysis (3rd edition) Theorem 7.15)

Therefore, $f := f^{(0,0)}$ is the candidate for the limit of $(f_m)_m$. But so far we only know $f^{(0,0)} \in C_b$. We need to show: $f \in C^\infty(\mathbb{R}^d)$ and $x^\alpha \partial_x^\beta f(x) = f^{\alpha,\beta}(x)$.

This would imply $f \in S(\mathbb{R}^d)$ and $d_S(f_m, f) \xrightarrow{m\to\infty} 0$, i.e., $f_m \xrightarrow{m\to\infty} f$ in S, and thus the completeness of (S, d_S).

Checking this in detail is a bit lengthy; let us here just show for $d=1$ that $f \in C^1(\mathbb{R}^d)$ and $\partial_x f = f^{(0,1)}$, the rest goes analogously.

Since $f_m \in S(\mathbb{R})$ $\forall m$, we have $f_m(x) = f_m(0) + \int_0^x f_m'(y)\,dy$.

Since $f_m \to f$ and $f_m' \to f^{(0,1)}$ <u>uniformly</u>, we can take the limit:

$$\lim_{m\to\infty} f_m(x) = f(x) = f(0) + \underbrace{\lim_{m\to\infty} \int_0^x f_m'(y)\,dy}$$

$$= \int_0^x f^{(0,1)}(y)\,dy \quad \text{due to uniform convergence}$$

Thus, $f \in C^1(\mathbb{R})$ and $f' = f^{(0,1)}$ $\square$

Next, we establish some standard properties of the Fourier transform on S.

Lemma 2.11: Properties of the Fourier transform

(1) $\forall \alpha, \beta \in \mathbb{N}_0^d,\ f \in \mathcal{S}$: $\quad \left((ik)^\alpha \partial_k^\beta \mathcal{F} f\right)(k) = \left(\mathcal{F} \partial_x^\alpha (-ix)^\beta f\right)(k)$,

in particular: $\widehat{(xf)}(k) = i(\nabla_k \hat{f})(k)$ and $\widehat{(\nabla_x f)}(k) = ik\hat{f}(k)$.

(2) $\mathcal{F}$ and $\mathcal{F}^{-1}$ are continuous linear maps $\mathcal{S} \to \mathcal{S}$.

Proof:

(1) Recall $(\mathcal{F} f)(k) = \hat{f}(k) = (2\pi)^{-\frac{d}{2}} \int f(x) e^{-ikx} dx$

Then with Lemma 2.2: $\left((ik)^\alpha \partial_k^\beta \mathcal{F} f\right)(k) = (2\pi)^{-\frac{d}{2}} (ik)^\alpha \partial_k^\beta \int e^{-ikx} f(x) dx$

note: all integrals exist since $f \in \mathcal{S}$

$$= (2\pi)^{-\frac{d}{2}} \int (ik)^\alpha (-ix)^\beta e^{-ikx} f(x) dx$$

$$= (2\pi)^{-\frac{d}{2}} (-1)^{|\alpha|} \int \left(\partial_x^\alpha e^{-ikx}\right) (-ix)^\beta f(x) dx$$

$|\alpha|$-times integration by parts (boundary terms vanish, since $f \in \mathcal{S}$)

$$= (2\pi)^{-\frac{d}{2}} \int e^{-ikx} \left(\partial_x^\alpha (-ix)^\beta f(x)\right) dx$$

$$= \mathcal{F}\left(\partial_x^\alpha (-ix)^\beta f\right)(k)$$

(2) next time