Last time we proved: (S, d_S) is complete.

Next: We continue our proof of

Lemma 2.11: Properties of the Fourier transform

(1) $\forall \alpha, \beta \in \mathbb{N}_0^d$, $f \in S$: $\left((ik)^\alpha \partial_k^\beta \mathcal{F} f\right)(k) = \left(\mathcal{F} \partial_x^\alpha (-ix)^\beta f\right)(k)$,

in particular: $\widehat{(xf)}(k) = i(\nabla_k \hat{f})(k)$ and $\widehat{(\nabla_x f)}(k) = ik\hat{f}(k)$.

(2) $\mathcal{F}$ and $\mathcal{F}^{-1}$ are continuous linear maps $S \to S$.

Proof: (1) was proved last time.

(2) On metric spaces continuity (preimages of open sets are open) is equivalent to sequential continuity ($d(f_n, f) \to 0$ implies $d(\mathcal{F}f_n, \mathcal{F}f) \to 0$).

Thus let us choose $f_n \to f$ in S, meaning $d_S(f_n, f) \to 0$, meaning $\|f_n - f\|_{\alpha,\beta} \to 0$ for all $\alpha, \beta \in \mathbb{N}_0^d$. We now show that $\|\mathcal{F}g\|_{\alpha,\beta} \le C \sum_{j=0}^{m} \sup_{|\tilde{\alpha}|+|\tilde{\beta}|=j} \|g\|_{\tilde{\alpha},\tilde{\beta}}$ for some $C > 0$ and $m \in \mathbb{N}$, which implies $\|\mathcal{F}f - \mathcal{F}f_n\|_{\alpha,\beta} \to 0$ $\forall \alpha, \beta \in \mathbb{N}_0^d$ if $\|f_n - f\| \to 0$ $\forall \alpha, \beta \in \mathbb{N}_0^d$.

We compute:

$$\|\mathcal{F}g\|_{\alpha,\beta} := \|k^\alpha \partial_k^\beta \mathcal{F}g\|_\infty$$

(1) and $|\int\ldots| \le \int|\ldots|$

$$\le (2\pi)^{-\frac{d}{2}} \int |\partial_x^\alpha x^\beta g(x)|\, dx$$

$$= (2\pi)^{-\frac{d}{2}} \int (1+|x|^2)^d \, |\partial_x^\alpha x^\beta g(x)| \, (1+|x|^2)^{-d} \, dx$$

$$\le (2\pi)^{-\frac{d}{2}} \left(\sup_{x\in\mathbb{R}^d} \left|(1+|x|^2)^d \partial_x^\alpha x^\beta g(x)\right| \right) \underbrace{\int (1+|x|^2)^{-d} \, dx}_{= \text{const.} \cdot \int_0^\infty (1+r^2)^{-d} r^{d-1} dr \,\le\, \tilde{C}}$$

(since integrand $\sim r^{-2d+d-1}$ for large r)

$$\le C \sum_{j=0}^{m} \sup_{|\tilde\alpha|+|\tilde\beta|=j} \|g\|_{\tilde\alpha,\tilde\beta} \quad \text{for some } m\in\mathbb{N},\ C>0.$$

$\square$

Theorem 2.12: $\mathcal{F}: \mathcal{S} \to \mathcal{S}$ is a continuous bijection with continuous inverse $\mathcal{F}^{-1}$.

Proof: HW: (1) Show $\mathcal{F}^{-1}\mathcal{F} = \mathrm{id}$ only on C_c^∞ = smooth fct.s with compact support

↳ consider $\operatorname{supp} f \subset [-m,m]^d$

$\Rightarrow$ Fourier series, write f as Riemann sum

(2) Show that C_c^∞ is dense in $\mathcal{S}$, then thm. follows from continuity.

↳ use some smooth cutoff function, e.g., $G(x) = \begin{cases} e^{-\frac{1}{1-x^2}+1} & \text{for } |x|<1 \\ 0 & \text{else.} \end{cases}$

Lemma 2.14: Plancherel on $\mathcal{S}$

For $f,g \in \mathcal{S}$, we have $\int \hat{f}(x) g(x)\, dx = \int f(x)\, \hat{g}(x)\, dx$, and, in particular,

$$\int |\hat{f}(k)|^2 dk = \int |f(x)|^2 dx.$$

<u>Proof:</u> simple computation, HW.